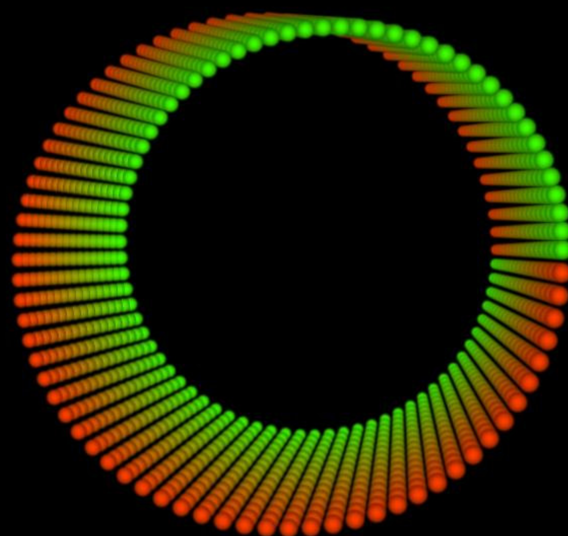
A - Level Core Pure Maths

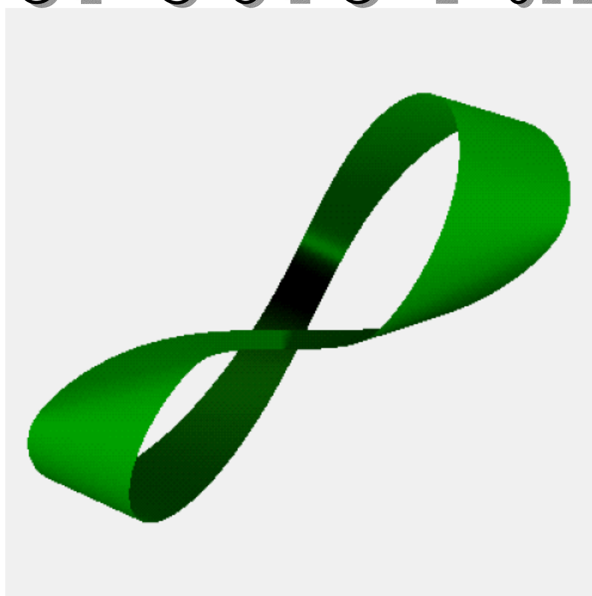


Email: mike_cook_1982@yahoo.co.uk

Version 1.0

Mike Cook

A - Level Core Pure Maths



Email: mike_cook_1982@yahoo.co.uk

Version 1.0

MIKE COOK

Preface

These notes may be downloaded and printed free of charge for personal and educational use only. Currently, the notes can be found at www.freewebs.com/mikecook The notes are loosely based on the current AQA A-level pure maths syllabus which can be found at the AQA website, www.aqa.org.uk The material found in the notes should be useful for people studying for exams set by other UK exam boards. I cannot guarantee the accuracy of the information contained in these notes; use them at your own risk! These notes are not intended to serve as a self-study course, they should be used under the guidance of a teacher. If you have found these notes useful for yourself or your students, let me know! If you spot any mistakes or have any suggestions or comments, you can email me mike_cook_1982@yahoo.co.uk

I hope that you find these notes useful. I wish you well in your studies.

The picture on the cover of these notes shows a Möbius strip. For more information, see the end of these notes.

Mike Cook, 2005

CONTENTS

Chapter 1

Chapter 2

Chapter 3

Chapter 4

GCE AS/A Mathematics Chapter 1

1.1 Algebra

Surds

A surd is a square root that cannot be expressed as a rational number (a quotient of real numbers), for example $\sqrt{2} = 1.414213562...$ is a surd

Rules of Surds

There are three important rules to remember when working with surds, they are:

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

Example 1.1

- $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$
- $\dfrac{\sqrt{24}}{\sqrt{8}} = \sqrt{\dfrac{24}{8}} = \sqrt{3}$
- $9\sqrt{5} - 4\sqrt{5} = (9-4)\sqrt{5} = 5\sqrt{5}$

Test 1.1 Simplify the following as much as possible using some or all of the three rules stated above:

$$\frac{20\left(\sqrt{3} \times \sqrt{3} \times \sqrt{9}\right) - 16\left(\sqrt{9} \times \sqrt{9}\right)}{\sqrt{8} \times \sqrt{2}}$$

Recall the first rule of surds above. We can use this rule in two ways, to multiply surds, but also to factorise surds. Factorising surds can lead to simplifications.

Example 1.2 $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$

When faced with a surd, look to see whether the surd can be factorised using square numbers (square numbers are 4, 9, 16, 25, 36, 49, 64, …), as we did in example 1.2.

Multiplying a Surd by a Real Number

Any real number, like 3, 7, 49 etc. can be expressed as a root by remembering that $a = \sqrt{a^2}$. For example, we can write $5 = \sqrt{5^2} = \sqrt{25}$. This means that we can multiply a surd by a real number, first by expressing the real number as a root, then using the first rule of surds.

! Do not get confused and write an incorrect statement, like $\sqrt{5} \times \sqrt{5} = 25$

Example 1.3 $\sqrt{512} - 15\sqrt{2} = \sqrt{2 \times 256} - 15\sqrt{2} = 16\sqrt{2} - 15\sqrt{2} = \sqrt{2}$

Test 1.2 Using any of the results so far, simplify the following as much as possible:

$\sqrt{75} + 2\sqrt{3}$

Rationalising the Denominator

When dealing with expressions where surds appear in the denominator, it is usual to eliminate all surds in the denominator where possible.

For example, given $\frac{2}{\sqrt{3}}$, we would eliminate the $\sqrt{3}$ term in the denominator by multiplying numerator and denominator by $\sqrt{3}$, i.e. $\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3}$. Of course, we now have a surd appearing in the numerator, but it is standard practice to favour surds in the numerator over surds in the denominator. This is an advantage, for example, if we wish to add various fractions involving surds.

To rationalise the denominator of $\frac{2}{2+\sqrt{5}}$, we multiply numerator and denominator by $2-\sqrt{5}$. This technique is an illustration of the factorisation of a *difference of squares*.

Recall: $(a+b)(a-b) = a^2 - b^2$.

Returning to our example, we have:

$$\frac{2}{2+\sqrt{5}}=\frac{2\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}=\frac{4-2\sqrt{5}}{4-5}=2\sqrt{5}-4$$

In general, to rationalise the denominator of $\frac{1}{\left(a\sqrt{b}\pm c\sqrt{d}\right)}$, we multiply numerator and denominator by $\left(a\sqrt{b}\mp c\sqrt{d}\right)$.

Example 1.4 Rationalise the denominator in the following expression:

$$\frac{2\sqrt{3}+\sqrt{2}}{3\sqrt{2}+\sqrt{3}}.$$

In this case, we can rationalise the denominator by multiplying numerator and denominator by $3\sqrt{2}-\sqrt{3}$. Let's do this:

$$\frac{2\sqrt{3}+\sqrt{2}}{3\sqrt{2}+\sqrt{3}}=\frac{\left(2\sqrt{3}+\sqrt{2}\right)\left(3\sqrt{2}-\sqrt{3}\right)}{\left(3\sqrt{2}+\sqrt{3}\right)\left(3\sqrt{2}-\sqrt{3}\right)}$$

$$=\frac{6\sqrt{6}-2\sqrt{9}+3\sqrt{4}-\sqrt{6}}{9\sqrt{4}-\sqrt{9}}$$

$$=\frac{6\sqrt{6}-2\times3+3\times2-\sqrt{6}}{9\times2-3}$$

$$=\frac{5\sqrt{6}-6+6}{15}$$

$$=\frac{\sqrt{6}}{3}$$

Test 1.3 Simplify the following expression as much as possible: $\frac{3+\sqrt{24}}{2+\sqrt{6}}$. (Start by rationalising the denominator).

Some More Examples

The general process for simplifying surds is to rationalise the denominator where appropriate, write the expression involving as few roots as possible and write the roots as small as possible, for example, we can write $\sqrt{112}$ as $4\sqrt{7}$ (Check!). Keep at the front of your mind all of the rules and methods we have covered.

Example 1.5 Simplify each of the following:

a) $5\sqrt{20}+2\sqrt{45}$ **b)** $\dfrac{3\sqrt{5}}{2\sqrt{6}}$

a) $5\sqrt{20}+2\sqrt{45}=5\sqrt{4\times5}+2\sqrt{5\times9}$

$$=5\times2\sqrt{5}+2\times3\sqrt{5}$$

$$=10\sqrt{5}+6\sqrt{5}$$

$$=16\sqrt{5}$$

b) $\dfrac{3\sqrt{5}}{2\sqrt{6}}=\dfrac{3\sqrt{5}\times\sqrt{6}}{2\times6}$

$$=\frac{3\sqrt{30}}{12}$$

$$=\frac{\sqrt{30}}{4}$$

Example 1.6 Express $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ in the form $a\sqrt{2}+b$ where a and b are integers.

The obvious thing to do here is to rationalise the denominator:

$$\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$$

$$=\frac{2+\sqrt{2}+\sqrt{2}+1}{2-1}$$

$=3+2\sqrt{2}$ So we see that $a=2$ and $b=3$.

Quadratic Functions and their Graphs

Introduction and revision

We are already familiar with the graphs of quadratic functions of the form $f(x)=ax^2+bx+c$. We can also factorise quadratic equations of this form where possible to find the roots (zeros) or use the quadratic formula for finding the roots when factorisation is not possible.

As a reminder, let us find the roots, and sketch, the following function: $f(x)=8x^2-14x+3$.

We know that the shape of this graph is a **parabola** (bucket shape). The parabola is the 'right war round' because the x^2 term is positive. Where does the parabola cross the x-

axis? We call the point(s) where the parabola crosses the *x*-axis the **roots** (or **zeros**) of the quadratic. These correspond the value(s) of *x* satisfying $8x^2 - 14x + 3 = 0$.

In this case we can factorise as follows:

$$8x^2 - 14x + 3 = 0$$

$$\Rightarrow \quad (4x-1)(2x-3) = 0$$

$$\Rightarrow \quad 4x-1=0 \quad \text{or} \quad 2x-3=0$$

$$\Rightarrow \quad x = \frac{1}{4} \quad \text{or} \quad x = \frac{3}{2}$$

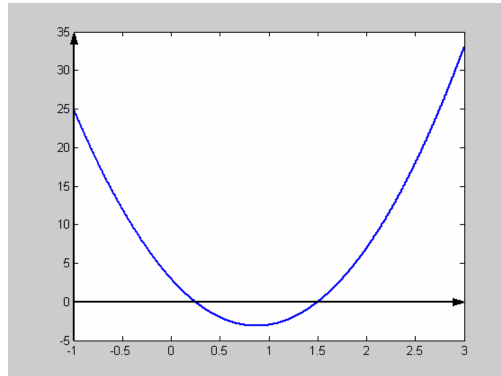


fig. 1.1

So the roots of the quadratic (the points where the quadratic graph crosses the *x*-axis) are at $x = \frac{1}{4}$ and $x = \frac{3}{2}$, as shown in *fig.1.1*.

The *minimum* of this graph occurs at $x = \frac{7}{8}$; we will learn how to calculate this in a later chapter. The point on the graph at where the minimum (or maximum in other cases) occurs is called the *vertex*. Notice that the graph has a *line of symmetry* - a vertical line that cuts through the vertex. Of course, this property is seen in all quadratic parabolas. This is illustrated in *fig1. 2.* on the simplest quadratic parabola - $f(x) = x^2$.

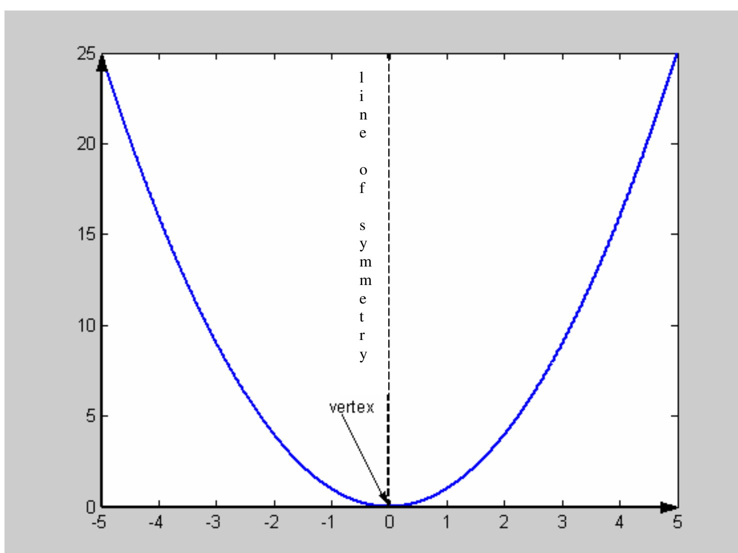


fig.1 2

Function Notation

Consider the function $f(x) = x^2$. What is the value of $f(x)$ when $x = 3$? The answer is, of course, 9. We write $f(3) = 9$ to say 'the value of the function *f* when $x = 3$ is 9'.

Factorisation of Quadratic Polynomials

When asked to find the roots of a quadratic polynomial, or to factorise a quadratic polynomial, try the following methods:

First, write the quadratic in the form $ax^2 + bx + c = 0$. For example, write $2x^2 = 7$ in the form $2x^2 - 7 = 0$. It may be necessary to do some manipulation first, for example clearing fractions where possible.

i. If there is only one term involving *x* – the equation can be solved by algebraic manipulation. For example, $3 = \frac{147}{x^2} \quad \Rightarrow \quad x^2 = \frac{147}{3} = 49 \quad \Rightarrow \quad x = \pm\sqrt{49} = \pm 7$

ii. If there is one term involving x^2 and one term involving *x* and no constant terms – then factorise out an *x*. For example,
$$x^2 = \frac{1}{2}x + 3x^2 \quad \Rightarrow \quad 2x^2 + \frac{1}{2}x = 0 \quad \Rightarrow \quad x\left(2x + \frac{1}{2}\right) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } 2x + \frac{1}{2} = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = -\frac{1}{4}$$

iii. If the quadratic contains x^2 and *x* and constant terms, try to factorise into two linear factors. For example
$$6x^2 = 12 - 6x \quad \Rightarrow \quad 6x^2 + 6x - 12 = 0 \quad \Rightarrow \quad (2x+4)(3x-3) = 0$$
$$\Rightarrow \quad 2x + 4 = 0 \text{ or } 3x - 3 = 0 \quad \Rightarrow \quad x = -2 \text{ or } x = 1$$

iv. If the quadratic does not factorise, use the quadratic formula. If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For example $3x^2 + 5x - 7 = 0$, then $x = \frac{-5 \pm \sqrt{5^2 - 4\times3\times(-7)}}{2\times3} = \frac{-5 \pm \sqrt{109}}{6}$ this is approximately equal to $x =$ 0.906 or $x = -2.573$.

Example 1.7 Find the (real) solutions of the following:

$$6x+6=\frac{12}{x}$$

This expression is a quadratic polynomial in disguise! First we need to do a bit of manipulation to get this into the standard form. Multiplying throughout by x to clear the fraction gives:

$$6x^2+6x=12 \quad \Rightarrow \quad 6x^2+6x-12=0$$

Now, this is *exactly the same* polynomial as before, we have not changed it in any way, just written it slightly differently. We are asked to find the solutions, i.e. find the values of x such that when we substitute in those values, we get a true statement. This is just a case of factorising and reading off the roots in the usual way. We factorise to get:

$$(2x+4)(3x-3)=0 \quad \Rightarrow \quad 2x+4=0 \text{ or } 3x-3=0$$

i.e. $x=-2$ or $x=1$.

Example 1.8 Find the (real) solutions of the following:

$$2x^4+x^3-6x^2=0$$

Now this may not look like a quadratic equation, but we can make it look like a quadratic equation by factoring out one of the x terms. Then we can factorise completely. Doing this gives:

$$2x^4+x^3-6x^2=0 \quad \Rightarrow \quad x^2\left(2x^2+x-6\right)=0 \quad \Rightarrow \quad x(2x-3)(x+2)$$

so we have that $x=0$ or $x=-2$ or $x=\frac{3}{2}$.

Example 1.9 Find the (real) solutions of the following:

$$m^4-10m^2+24=0$$

Here, factorising out x^2 will not work so well, because of the constant term. We can, however, make this look like a quadratic equation by setting $a=m^2$. Making this simple substitution gives:

$$a^2-10a+24=0.$$

We can factorise this easily as $(a-4)(a-6)=0$ so that $a=4$ or $a=6$. But, remember that $a=m^2$ - the original question was in terms or *m*, not *a*, so we must give our answer

in terms of m. So, $m^2 = 4$ or $m^2 = 6$ $\Rightarrow$ $m = \pm 2$ or $m = \pm\sqrt{6}$. (Don't forget $\pm$).
These four roots are shown in *fig.1.3*

$$f(x) = m^4 - 10m^2 + 24$$

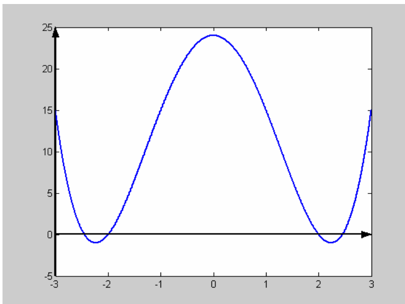


fig. 1.3

> **Test 1.3** Solve each of the following for x:
>
> **a)** $2x + 7 = \dfrac{4}{x}$ **b)** $x^2 - x = 0$
>
> **c)** $x + 7 = \dfrac{6}{2x - 2}$

Completing the Square

Consider the function $f(x) = x^2 - 6x + 9$. This can be factorised as $f(x) = (x-3)(x-3) = (x-3)^2$, i.e. it is a perfect square.

What about the function $f(x) = x^2 - 6x + 10$ can this be factorised as a perfect square? The answer is, of course, no. But we can write it as $f(x) = (x-3)^2 + 1$.

What about $g(x) = x^2 - 4x + 2$. Can we write this in the form $g(x) = (x+a) + b$? Yes, we can: $g(x) = (x-2)^2 - 2$ ($a = -2,\ b = -2$). Can we write any quadratic polynomial of the form $h(x) = x^2 + bx + c$ in the form $h(x) = (x+A)^2 + B$? The answer, once again, is yes. In general, this technique is called **completing the square**. Before we construct a general method, let us first think carefully about what we are doing when we expand an expression like $(x+a)^2$.

$$(x+a)^2 = x^2 + 2ax + a^2$$

- The first term (x) gets squared
- The two terms (x and a) get multiplied together and doubled
- The last term (a) gets squared

Completing the square is essentially the *reverse* of this process.

<u>METHOD</u>
To complete the square of $x^2 + bx + c$, we write,

$$x^2 + bx + c = (x + A)^2 + B.$$

<u>To find A:</u> Half the coefficient of *x*, i.e. $A = \frac{1}{2}b$

<u>To find B</u>: expand the bracket and see what needs to be added or subtracted to get equality.

We have established that $A = \frac{1}{2}b$, so we have $x^2 + bx + c = \left(x + \frac{1}{2}b\right)^2 + B$.

Now $\left(x + \frac{1}{2}b\right)^2 = x^2 + bx + \frac{1}{4}b^2$, So we have

$$x^2 + bx + c = x^2 + bx + \frac{1}{4}b^2 + B \qquad \Rightarrow \quad B = c - \frac{1}{4}b^2$$

<u>SUMMARY:</u> $x^2 + bx + c = (x + A)^2 + B$ where $A = \frac{1}{2}b$ and $B = c - \frac{1}{4}b^2$.

It is better to remember the method, rather than memorise the result.

Example 1.10 Express $x^2 + 5x - 7$ in the form $(x + A)^2 + B$ and hence solve the equation $x^2 + 5x - 7 = 0$.

We complete the square, $x^2 + 5x - 7 = \left(x + \frac{5}{2}\right)^2 - 7 - \frac{25}{4} = \left(x + \frac{5}{2}\right)^2 - \frac{53}{4}$.

Now, $x^2 + 5x - 7 = 0 \quad \Rightarrow \quad \left(x + \frac{5}{2}\right)^2 - \frac{53}{4} = 0$

$$\Rightarrow \quad x + \frac{5}{2} = \pm\sqrt{\frac{53}{4}} \quad \Rightarrow \quad x = \pm\sqrt{\frac{53}{4}} - \frac{5}{2}$$

or $x = 1.14$ or $x = -6.14$.

So far, we have only completed the square for expressions of the form $x^2 + bx + c$. What about expressions of the form $ax^2 + bx + c$? The solution to this problem is to factor out the *a* first. For example, if we want to complete the square for $3p^2 - 7p + 8$, we first factor out the 3 to get $3\left[p^2-\frac{7}{3}p+\frac{8}{3}\right]$. Now we complete the square of $p^2-\frac{7}{3}p+\frac{8}{3}$ in the usual way, but remember that everything is multiplied by 3.

Example 1.11 Express $2x^2-3x+1$ in the form $A(x+B)^2+C$.

$$2x^2-3x+1=2\left[x^2-\frac{3}{2}x+\frac{1}{2}\right]$$

$$=2\left[\left(x-\frac{3}{4}\right)^2-\frac{9}{16}+\frac{1}{2}\right]$$

$$=2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]$$

! Be careful with the brackets here, make sure everything gets multiplied by the 2.

$$=2\left(x-\frac{3}{4}\right)^2-\frac{1}{8}$$

i.e. $A=2,\quad B=-\frac{3}{4},\quad C=-\frac{1}{8}$.

Test 1.4 Express $3x^2-x+10$ in the form $A(x+B)^2+C$ and hence show that the equation $3x^2-x+10=0$ has no real root.

Minimum Value of a Function

Why is completing the square useful? As we have seen in example 10, we can solve a quadratic equation by completing the square, although in practice quadratic equations are not normally solved in this way. However, a generalisation of this method is used to prove the quadratic equation formula (see later).

We can gain a useful piece of information by completing the square of a quadratic equation, that is we can say what the **minimum value** of the quadratic is. Once we have expressed a quadratic in the form $A(x+B)^2+C$, since $(x+B)^2\geq 0$ (anything squared is never negative, i.e. always greater than or equal to zero), the minimum value must be *C*. The minimum (or maximum) point of a quadratic function is called the **turning point** of the function.

Example 1.12 Find the turning point (x, y) on the function $y = x^2 + 16x + 63$.

This function is a parabola (right way round), so the turning point is a **minimum** turning point. We can find this by completing the square:

$$x^2 + 16x + 63 = (x+8)^2 - 1$$

Since the minimum value of $(x+8)^2$ is 0, the minimum value of $y = x^2 + 16x + 63$ occurs at $y = -1$.
We find the *x* value at this point by substituting $y = -1$ into the equation and solving for *x*:

$$x^2 + 16x + 63 = -1 \quad \Rightarrow \quad x^2 + 16x + 64 = 0 \quad \Rightarrow \quad (x+8)^2 = 0 \quad \Rightarrow \quad x = -8.$$

So the minimum point occurs at $(-8, -1)$, as shown in *fig.1. 4*.

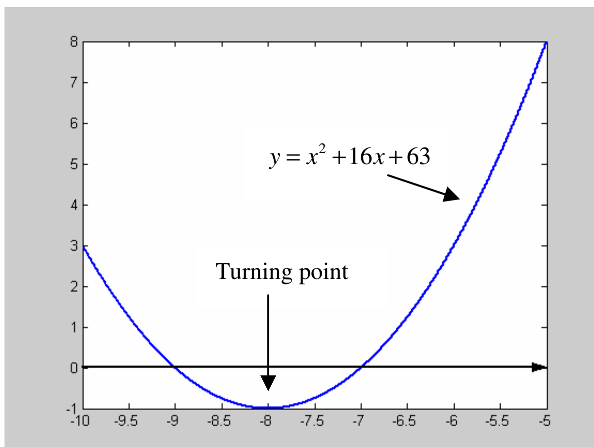


fig. 1.4

The Quadratic Formula

Recall the quadratic formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This can be proved by completing the square of $ax^2 + bx + c$ and using it to solve $ax^2 + bx + c = 0$. This is left as an exercise.

The Discriminant of a Quadratic Equation

Look back at example 1.7. Here we considered a quadratic equation which had **two distinct** (different) **roots**, namely $x = -2$ or $x = 1$.

Look back at example 12. Here we came across the quadratic $x^2+16x+64=0$, which we found had only **one root** (or sometimes we say two repeated roots).

Now consider the quadratic $x^2+x+6=0$. This will not factorise. If we try to use the quadratic formula we get:

$x=\dfrac{-1\pm\sqrt{1-18}}{2}=\dfrac{-1\pm\sqrt{-23}}{2}$. This presents us with a problem – we do not know how to find the square root of a negative number. So we say that this quadratic has **no** (real) **roots**.

In summary, any quadratic equation $ax^2+bx+c=0$ has either: two different roots, one root (two repeated roots) or no (real) roots.

Given any quadratic equation $ax^2+bx+c=0$, can we tell which category it will fall into without going through the whole process of solving the equation?

Recall, once again the quadratic formula: If $ax^2+bx+c=0$, then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.
It turns out that the information about whether the quadratic has two, one or no root(s) is contained in the term b^2-4ac. This term is called the **discriminant** of the quadratic equation.

In the case where the quadratic has two distinct roots, the equation cuts the *x*-axis in two different places. In the case where the quadratic has one root, the equation is **tangent** to the *x*-axis at one point (just touches but does not cross). In the case where the quadratic has no (real) roots, the equation never crosses the *x*-axis.

- If $b^2-4ac>0$ then the equation has two distinct roots (*fig. 1.5)*
- If $b^2-4ac=0$ then the equation has one root (two repeated roots) (*fig.1. 6)*
- If $b^2-4ac<0$ then the equation has no (real) roots (*fig.1. 7)*

$b^2-4ac>0$ - crosses x-axis twice

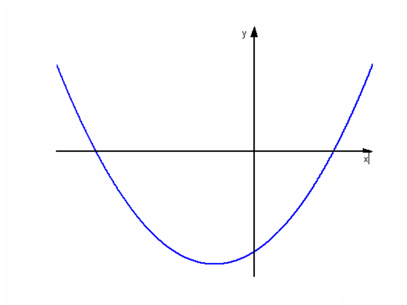


fig. 1.5

$b^2-4ac=0$ - tangent to x-axis

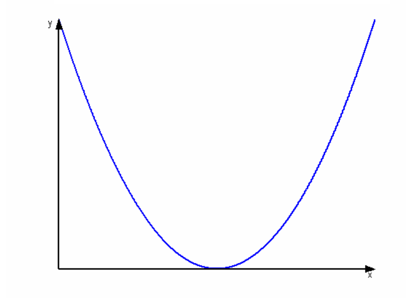


fig. 1.6

$b^2-4ac<0$ - does not cross x-axis

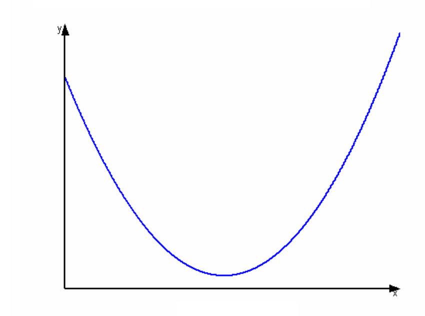


fig. 1.7

Example 1.13 Does the equation $\frac{1}{2}x^2+4x+8=0$ have any roots? If so, does it have two repeated roots or two distinct roots? Which of *fig. 1.5, 1.6, 1.7* could be a plot of $y=\frac{1}{2}x^2+4x+8$?

We need to calculate b^2-4ac. $b^2-4ac=4^2-4\times\frac{1}{2}\times8=0$, so $\frac{1}{2}x^2+4x+8=0$ has two repeated roots. $y=\frac{1}{2}x^2+4x+8$ is tangent to the x-axis at one point, so *fig. 1.6* could be a plot of $y=\frac{1}{2}x^2+4x+8$.

Test 1.5 **a)** Does $p^2-5p+6=0$ have two distinct, two repeated or no (real) roots?
b) Given that $ax^2+bx+c=0$ has two repeated roots, does $2ax^2-4bx+c=0$ have two distinct, two repeated or no (real) roots?

Curve Sketching

Quadratic functions

If we are asked to sketch a quadratic equation, there are four basic pieces of information we need to know: 'which way round' the quadratic equation is, depending on whether the x^2 term is positive or negative, where the graph crosses the x-axis (if at all), where the minimum (or maximum) value occurs and where the graph cuts the y-axis. To sketch a given quadratic, we first find the roots to see where it will cross the x-axis (if at all). Then we express the quadratic in the form $A(x+B)^2+C$, so that we can find the coordinates of the minimum value. To find where the graph crosses the y-axis, se simply set $x=0$ and calculate the value of y.

Example 1.14 Sketch the function $y = 2x^2 + 4x - 6$.

To find the roots, we factorise $2x^2 + 4x - 6 = 0$ to get:
$(2x+6)(x-1) = 0 \quad \Rightarrow \quad x = -3 \quad \text{or} \quad x = 1$.

So the $y = 2x^2 + 4x - 6$ cuts the *x*-axis at $x = -3$ and $x = 1$. Next we find the minimum point of the graph. We can write $2x^2 + 4x - 6 = 0$ as $2(x+1)^2 - 8 = 0$ (check). Therefore, the minimum value of $y = 2x^2 + 4x - 6$ occurs when $y = -8$. To find the *x* coordinate of the minimum point, substitute $y = -8$ into $y = 2x^2 + 4x - 6$ to get:

$$2x^2 + 4x + 2 = 0 \quad \Rightarrow \quad (2x+2)(x+1) = 0 \quad \Rightarrow \quad x = -1.$$

So the minimum point is at $(x, y) = (-1, -8)$.

To find where the graph crosses the *y*-axis, we set $x = 0$, to get $y = -6$.

Recall also that there is a line of symmetry parallel to the *y*-axis that cuts through the minimum point.

We now have all the information we need to sketch a plot. The plot is shown in *fig.1. 8*.

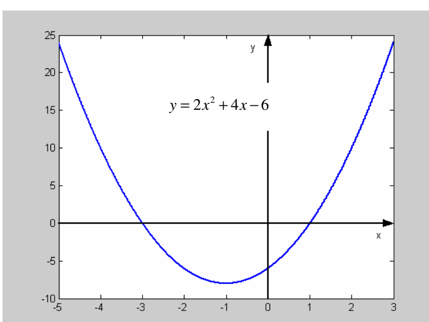


fig. 1.8

Test 1.6 Sketch the function $y = x^2 - 4x + 6$.

Simultaneous Equations

We are already familiar with solving two linear simultaneous equations in two unknowns. Recall that there are two methods of solving simultaneous linear equations: by elimination or by substitution. The elimination method relies on us being able to rewrite the equations so that after we add the two equation or subtract the two equations, one of the unknowns is eliminated. We can then solve the problem.
The substitution method relies on us being able to rearrange one of the equations to make one of the unknowns the subject. We then substitute for this unknown into the other equation and solve. Let us remind ourselves with some examples.

Example 1.15 Solve the following simultaneous equations:

$2x+6y=16$ ....(1)

$x-18y=-49$ ....(2)

Here, we can use the elimination method. We can eliminate either *x* or *y*. We will choose to eliminate *x*.

Multiplying **(2)** by 2 and subtracting the two equations gives:

$42y=114 \quad \Rightarrow \quad y=\frac{114}{42}=\frac{19}{7}=2.714$ We can now substitute for *y* in either **(1)** or **(2).**

Substituting for *y* in **(1)** gives:

$$2x+6\times\frac{19}{7}=16 \quad \Rightarrow \quad 2x=16-\frac{114}{7} \quad \Rightarrow \quad x=\frac{1}{2}\left(16-\frac{114}{7}\right)$$

$$\Rightarrow \quad x=-0.1429$$

Example 1.16 Solve the following simultaneous equations:

$x^2+y^2=13$ ....(1)

$2x+y=7$ ....(2)

The best way to solve these is to rearrange **(2)** in terms of either *x* or *y* and then substitute into **(1)**. We will rearrange **(2)** for *y*.

From **(2)** we have:

$y=7-2x$ ....(3)

Now we can substitute this into **(1)** and solve the resulting equation for *x*.

Substituting into **(1)** gives:

$$x^2+(7-2x)^2=13 \quad \Rightarrow \quad x^2+49-28x+4x^2=13 \quad \Rightarrow \quad 5x^2-28x+36=0$$

We can factorise this quadratic as: $(5x-18)(x-2)=0 \quad \Rightarrow \quad x=\frac{18}{5}$ or $x=2$

Now we substitute for *x* in **(3)**:

If $x=\frac{18}{5}$, then $y=7-2\times\frac{18}{5}=-\frac{1}{5}$

If $x=2$, then $y=7-2\times2=3$

Substitute these answers back into **(1)** and **(2)** to check that **both** pairs are valid solutions.

What do these results correspond to graphically? We have solved *simultaneously* $x^2 + y^2 = 13$ and $2x + y = 7$. The points $(x, y) = \left(\frac{18}{5}, -\frac{1}{5}\right)$ and $(x, y) = (2,3)$ correspond to points where the two graphs are simultaneously equal, i.e. where they have the same value, i.e. where they cross. This is illustrated in *fig. 1.9*.

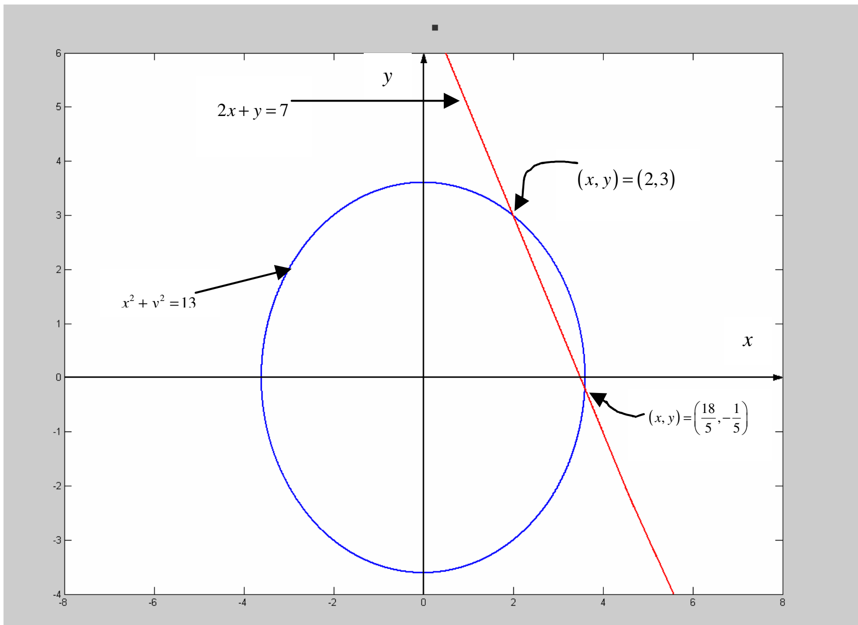


fig. 1.9

Test 1.7 Solve the following simultaneous equations. Illustrate the results graphically.

$y = x^2 - 2x + 2$

$y = 4x - 7$.

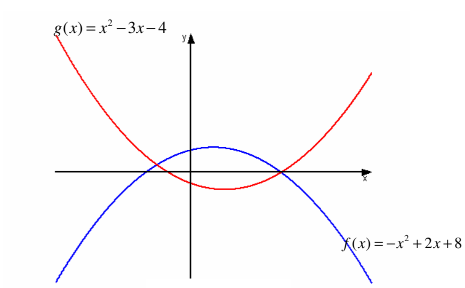


fig. 1.10

Test 1.8 Solve the simultaneous equations $2x^2 - xy + y^2 = 32$ and $y = -\frac{5}{x}$

Test 1.9 *fig. 1.10* shows plots for the functions $f(x) = -x^2 + 2x + 8$ and $g(x) = x^2 - 3x - 4$. Find the coordinates of the two points where the graphs of $f(x)$ and $g(x)$ cross.

Inequalities

Linear Inequalities

An **inequality** is an expression similar to an equation, but rather than having an equals sign, we have an inequality sign. An example of an inequality is, $2x+3\geq 11$. We can solve and manipulate inequalities in a similar way as we do with equations. We can add an amount to both sides, divide both sides by an equal amount or multiply both sides by an equal amount, as we do with equations. However, there is one golden rule we must always remember when working with inequalities:

! If we multiply or divide both sides of an inequality by a negative number, then we must reverse the inequality sign.

This can be easily illustrated. Consider the inequality, $6<8$. This is a true statement. If we multiply both sides by -1 we get, $-6<-8$, which is **false**. Because we have multiplied both sides by a negative number, we must reverse the inequality sign to get $-6>-8$, which is true. If we divide both sides by -2, for example, we must again reverse the inequality sign to get a true statement (check). So, while we can multiply and divide both sides of an equation by a negative number without worry, **when multiplying or dividing both sides of an inequality by a negative number, we must reverse the inequality sign**.

Example 1.17 Simplify the inequality $2x+3\geq 11$.

In this example, we work just as we would if this were an equation. First subtract 3 from both sides:

$2x\geq 8$. Then divide both sides by 2:

$x\geq 4$. So we have discovered that $2x+3\geq 11 \quad\Leftrightarrow\quad x\geq 4$, i.e. if we substitute any number greater than or equal to 4 for x in $2x+3\geq 11$, we will get a true statement.

Example 1.18 Simplify the inequality $\frac{1}{3}x<3x-4$.

Again, this is very similar to how we solve linear equations. First, multiply both sides by 3 to get $x<9x-12$. Now subtract $9x$ from both sides to get $-8x<-12 \quad\Rightarrow\quad x>\frac{3}{2}$.

Example 1.19 Find the set of integers which satisfy simultaneously both of:

$6x-3\leq 7(x-1)$ ....**(1)**

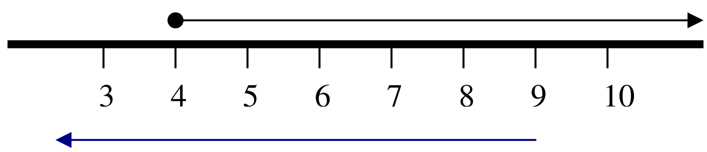
$2x > 3x - 9$ ……………....**(2)**

We start by simplifying each of **(1)** and **(2)** separately as normal. Simplifying **(1)** leads to:

$4 \le x$ or $x \ge 4$ (check). Simplifying **(2)** leads to:

$9 > x$ or $x < 9$.

Now we are asked for integers that simultaneously satisfy both **(1)** and **(2)**. The integers that satisfy **(1)** and **(2)** separately are illustrated in *fig1. 11*.



Note: 4 is included in the range, 9 is not.

fig. 1.11

So the integers which satisfy both inequalities simultaneously are 4, 5, 6, 7, 8.

Test 1.10 Find the range of values which satisfy simultaneously both of:

$7x \le 16 + 2x$ and

$\frac{1}{3}(3x+3) > 0$

Note: you are asked for a range of values here, not just integers.

Quadratic Inequalities

When solving quadratic inequalities, it is always advisable to make a sketch to see what is going on. Let us consider an example.

Example 1.20 Solve the inequality $8x^2 + 24x + 10 < 0$.

The way to solve this is to simply sketch the graph of $y = 8x^2 + 24x + 10$ and read off the values. We can factorise $8x^2 + 24x + 10$ as $(4x+2)(2x+5)$, so $y = 8x^2 + 24x + 10$ cuts the x-axis at $x = -\frac{1}{2}$ and $x = -\frac{5}{2}$. We can now make a rough sketch of the graph. The plot is shown in *fig. 11*.

Now, the region $8x^2 + 24x + 10 < 0$ corresponds to the region of the graph below the *x*-axis, i.e. in the region $-\frac{5}{2} < x < -\frac{1}{2}$.

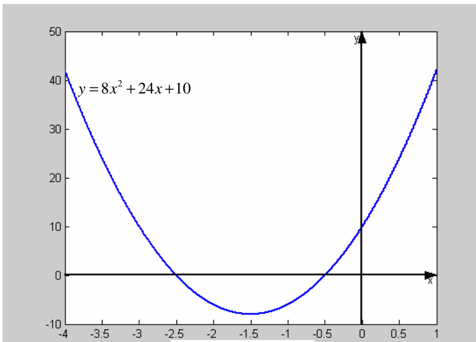


fig. 1.12

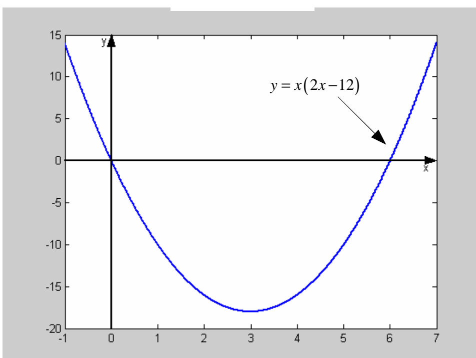


fig. 1.13

Example 1.21 Solve the inequality $2x^2 > 12x$.

We can rearrange $2x^2 > 12x$ to get $x(2x-12) > 0$. Now we plot $y = x(2x-12)$ and read off the answer. The plot is shown in *fig.1. 13*.

The region $x(2x-12) > 0$ corresponds to $x < 0$ and $x > 6$

Another way to illustrate this inequality is to work out where the graphs of $y = 2x^2$ and $y = 12x$ cross. We can plot these two graphs and write down the required range of values of *x* that satisfy the inequality. The plots of $y = 2x^2$ and $y = 12x$ are shown in *fig.1. 14*.

The region $2x^2 > 12x$ corresponds to the region where the graph of $y = 2x^2$ is above the graph of $y = 12x$, i.e. the regions $x < 0$ and $x > 6$

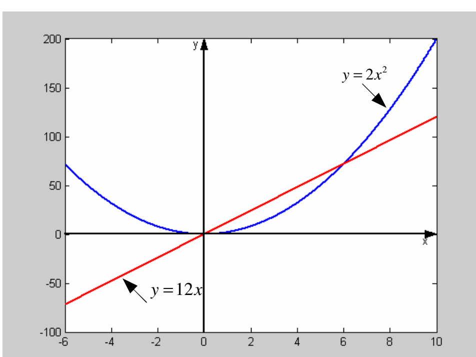


fig. 1.14

Example 1.22 Solve the inequality $\frac{8}{x+3} \le 2$.

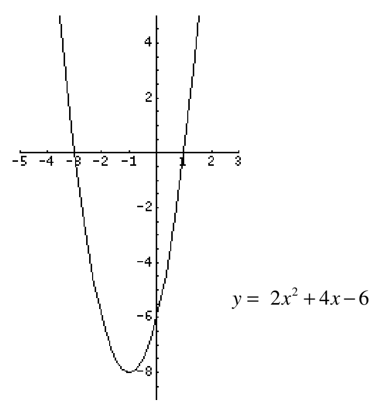


fig. 1.15

First we must clear the fraction. Beware: we cannot multiply both sides by $x+3$, because this may be a negative quantity. To ensure we are multiplying both sides by a positive quantity, we multiply by $(x+3)^2$. Doing this gives,

$8(x+3) \le 2(x+3)^2 \quad \Rightarrow \quad 2x^2+4x-6 \ge 0$ (check!)

$\Rightarrow \quad (2x+6)(x-1) \ge 0$.

So, $\dfrac{8}{x+3} \le 2 \quad \Leftrightarrow \quad x \le 3 \quad \text{or} \quad x \ge 1$

> **Test 1.11** Solve the inequality $\dfrac{2x+3}{x-1} \le 3$ for *x*.

The Remainder and Factor Theorems

Algebraic Division

In this section, our aim is to understand how to divide a quadratic or cubic polynomial by a linear term. For example, how do we work out $3x^2+12x+9$ divided by $x+2$? When we divide a quadratic by a linear expression, we expect the answer to be linear (When we divide an expression of order *m* by an expression of order *n*, we expect the answer to be an expression of order $m-n$). Before we tackle the problem of algebraic division, it will help us to first recall how we divide numbers. Consider 32 divided by 5. We can write:

$\dfrac{32}{5} = 6$ remainder 2. Alternatively we can write $32 = 6\times5+2$.

Now, when faced with $\dfrac{3x^2+12x+9}{x+2}$ we can write:

$3x^2+12x+9 \equiv (\text{linear term in } x)(x+2)+\text{remainder}$.

The remainder, in this case will be a constant, one degree less that the 'linear term'. We write the linear term in the general form $ax+b$:

$3x^2+12x+9 \equiv (ax+b)(x+2)+r$.

Notice, we use the symbol '$\equiv$' instead of an equals sign. This sign means 'identically equal to' and is used when the expression is valid for *all* values of *x*. Note: identities cannot be solved like equations can.

So the expression on the left of the identity sign is exactly the same as the expression on the right of the identity sign. Because we have 3 lots of x^2 on the left of the identity sign, we must have 3 lots of x^2 on the right of the identity sign. Look at the expression on the RHS; the only place where we will get x^2 terms is when we multiply ax by x:

$$(ax+b)(x+2)$$

i.e. we will have *a* lots of x^2 terms on the RHS. This means that $a=3$. Next consider the *x* terms. On the LHS we have 12 lots of *x*. On the RHS we will get *x* terms when we multiply ax by 2 and when we multiply b by *x*:

$$(ax+b)(x+2)$$

i.e. we will have $2a+b$ lots of *x* on the RHS, so $12=2a+b$. Since $a=3$ we have that $b=6$. So now we can write:

$3x^2+12x+9 \equiv (3x+6)(x+2)+r$. Multiplying out the brackets, we can see that $r=-3$.

Finally we can write down the answer to the original problem:

$$\frac{3x^2+12x+9}{x+2} \equiv 3x+6-\frac{3}{x+2}.$$

From $\dfrac{3x^2+12x+9}{x+2}$ we write, $3x^2+12x+9 \equiv (3x+6)(x+2)-3$.

The term $(3x+6)$ is sometimes called the **quotient**. We also say that $(3x+6)$ is a **factor** of $3x^2+12x+9$. -3 is the **remainder**.

The Remainder Theorem

When the polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$

This can be proved by the following argument.

Write $f(x)=(x-a)(\text{Quotient})+\text{Remainder}$. When $x=a$,

$f(a)=(a-a)(\text{Quotient})+\text{Remainder}$, i.e. $f(a)=\text{Remainder}$ as stated.

Example 1.23 Find the remainder when $f(x)=3x^3+4x^2+x+6$ is divided by $x-3$.

From the remainder theorem, we have that the required remainder is:

$f(3)=3\times3^3+4\times3^2+3+6=126$.

The following is a corollary of the remainder theorem:

$f(a)=0 \quad \Leftrightarrow \quad (x-a)$ is a factor of $f(x)$

This is often used when factorising cubic equations, as illustrated in the next example.

Example 1.24 Write $f(x)=x^3+4x^2+x-6$ as a product of three linear factors.

The aim is to find a number a such that $f(a)=0$, this will give us one of the factors, namely $(x-a)$. Usually, trying the numbers $\pm1,\ \pm2,\ \pm3$ will reveal at least one of the factors. Here we notice that $f(1)=0$. This means that $(x-1)$ is a factor of $f(x)$. Now we can write:

$f(x)=(x-1)(\text{Quadratic factor})$. We write the quadratic factor generally as ax^2+bx+c,

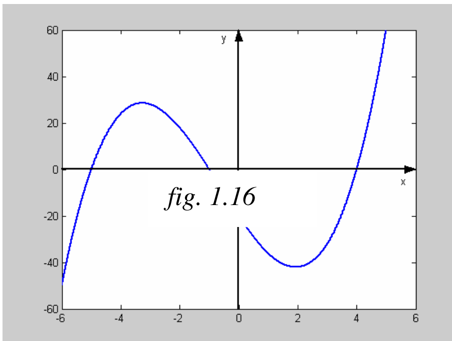
$f(x)=(x-1)(ax^2+bx+c)$. Comparing coefficients of x^2, x and constant terms reveals that $a=1,\ b=5,\ c=6$, i.e.

$f(x)=(x-1)(x^2+5x+6)$. The quadratic term can be factorised to give

$f(x)=(x-1)(x+2)(x+3)$. We have factorised $f(x)$ as a product of three linear factors as required. We may have also noticed that $f(-2)=0$ and $f(-3)=0$, this would have given us the three linear factors immediately, though not all examples are as obvious as this.

Test 1.12 Find the remainder when $g(m)=2m^3-5m^2-37m+60$ is divided by $x-4$. Express $g(m)$ as a product of three linear factors.

Test 1.13 Suggest a possible equation for the curve in *fig. 1.16*.



Hint: Look at where the graph cuts the *x*-axis

Graph transformations

Sketch the graph of $f(x) = x^2$. On the same graph, sketch $f(x) + 3 = x^2 + 3$ and $f(x) - 3 = x^2 - 3$. We can see that the graphs all have the same basic shape, but they are 'shifted' parallel to the *y*-axis. $f(x) = x^2$ has a minimum at $y = 0$. $f(x) + 3 = x^2 + 3$ has a minimum at $y = 3$, this is because every point on the graph of $f(x) = x^2$ has had 3 added to it. $f(x) - 3 = x^2 - 3$ has a minimum at $y = -3$, this is because every point on the graph of $f(x) = x^2$ has had 3 subtracted from it. We call such a 'shift', where the shape of the graph remains the same but the graph is moved relative to the axis, a **translation**. The plots are shown in *fig. 1.17*.

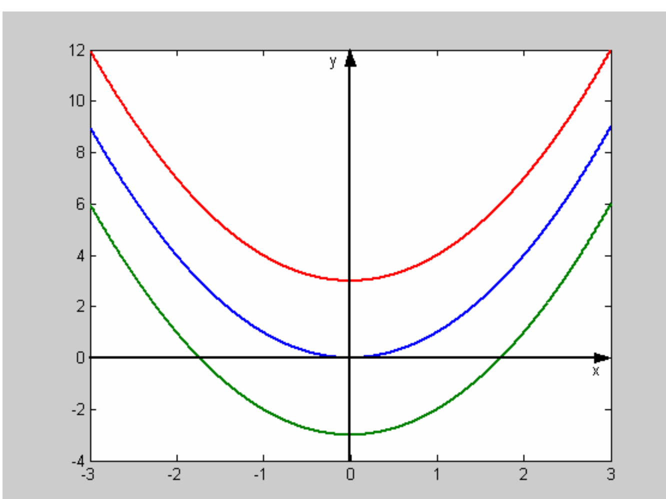


fig.1. 17

In general we can say that:

The transformation $f(x) \rightarrow f(x) + a$ is a translation by a units parallel to the *y*-axis in the positive direction.

The transformation $f(x) \rightarrow f(x) - a$ is a translation by a units parallel to the *y*-axis in the negative direction.

Sketch the graph of $f(x) = x^2$. On the same graph, sketch $f(x-2) = (x-2)^2$ and $f(x+2) = (x+2)^2$. What are the relationships between the three graphs? Note, $f(x-2)$ means 'substitute $x-2$ for *x* in the expression $f(x) = x^2$' and $f(x+2)$ means 'substitute $x+2$ for *x* in the expression $f(x) = x^2$'. The plots are shown in *fig. 1.18*.

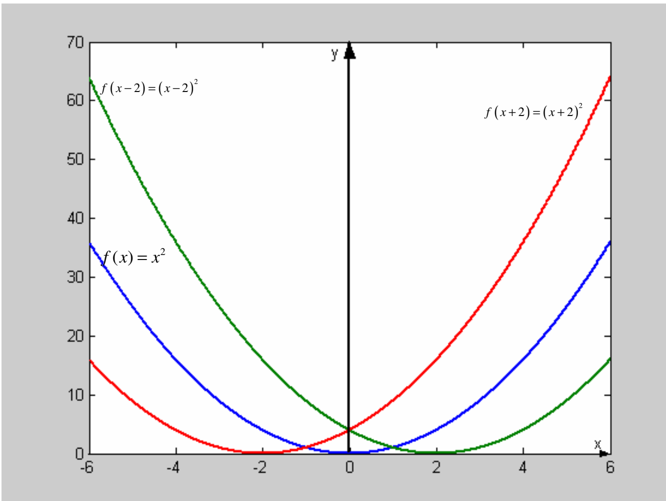


fig. 1.18

We can see that the graphs all have the same basic shape, but they are translations of each other parallel to the *x*-axis. $f(x) = x^2$ has a minimum at $x = 0$. $f(x-2) = (x-2)^2$ has a minimum at $x = 2$, i.e. it is a translation of $f(x)$ by 2 units parallel to the *x*-axis in the positive direction. $f(x+2) = (x+2)^2$ has a minimum at $x = -2$, i.e. it is a translation of $f(x)$ by 2 units parallel to the *x*-axis in the negative direction.

In general, we can say that:

The transformation $f(x) \rightarrow f(x-a)$ is a translation by *a* units parallel to the *x*-axis in the positive direction.

The transformation $f(x) \rightarrow f(x+a)$ is a translation by *a* units parallel to the *x*-axis in the negative direction.

!

This seems counter intuitive. The translation $f(x) \rightarrow f(x-a)$ moves the graph in the **positive direction** and the translation $f(x) \rightarrow f(x+a)$ moves the graph in the **negative direction**. The signs may cause us confusion.

Sketch the graph of $f(x) = x^2$. On the same graph sketch $3f(x) = 3x^2$. What is the relationship between the two graphs? The plots are shown in *fig.* 1.*19*.

Here, the basic shape of the graph has changed. In fact, the graph of $f(x)=x^2$ has been stretched parallel to the y-axis by a factor of 3 to make the graph of $3f(x)=3x^2$.

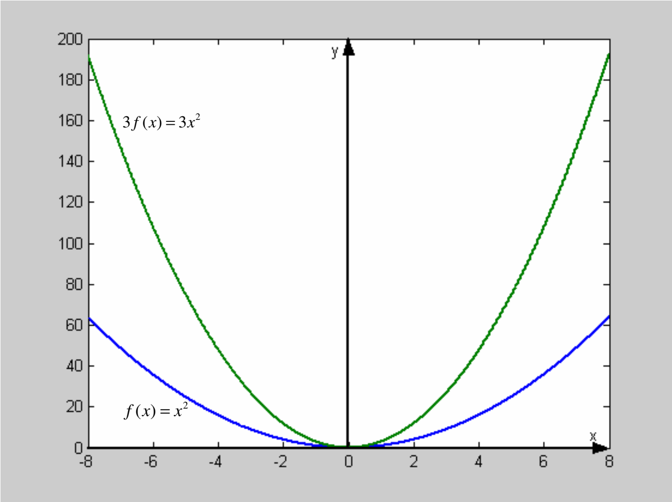


fig. 1.19

In general we can say that:

The transformation $f(x) \rightarrow af(x)$ is a stretch, parallel to the y-axis, by a factor a.

Note: Points $(x, y)=(0, y)$ are unaffected by this transformation. The same is true for the following type of transformation.

Sketch the graph of $f(x)=x^2$. On the same graph sketch $f(2x)=4x^2$. What is the relationship between the two graphs? The plots are shown in *fig.* 1.*20*.

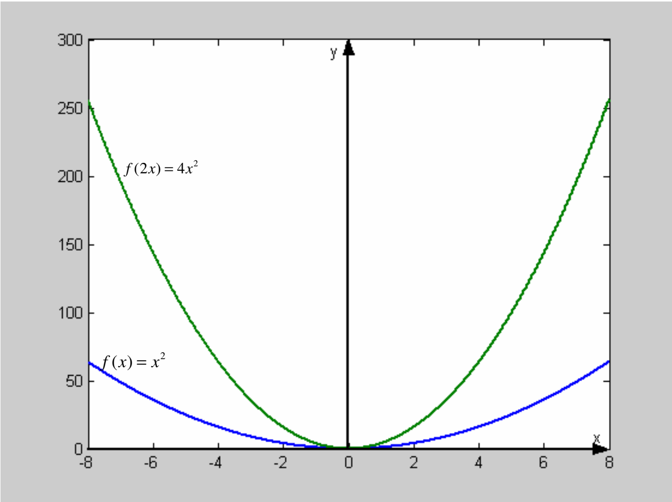


fig.1. 20

Here, the graph of $f(x)=x^2$ has been stretched parallel to the x-axis by a factor of $\frac{1}{2}$ to produce the graph of $f(2x)$. (We may think of this as a 'squash' parallel to the x-axis).

In general we can say that:

The transformation $f(x) \rightarrow f(ax)$ is a stretch, parallel to the *x*-axis, by a factor $\frac{1}{a}$.

Example 1.25 Sketch the function $f(x) = x^3$. On the same graph, sketch $g(x) = 2x^3 - 3$.

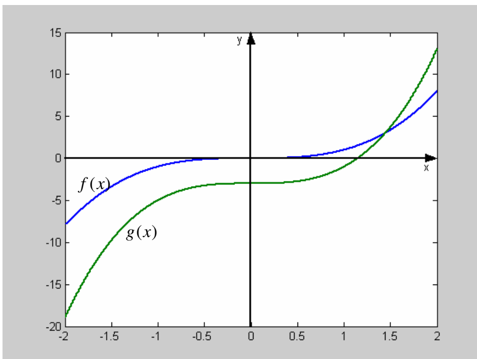


fig. 1.21

We first sketch the graph of $f(x) = x^3$. We perform two transformations on this graph to get the graph of $g(x) = 2x^3 - 3$. We notice that $g(x) = 2f(x) - 3$. The effect of multiplying $f(x)$ by 2 is to stretch the graph by a factor of 2 parallel to the *y*-axis. The effect of subtracting 3 from $f(x)$ translates the graph parallel to the *y*-axis 3 units in the negative direction. Combining these transformations leads to the plot shown in *fig.1. 21*.

Test 1.14 The function *f* is defined by $f(x) = \frac{1}{2(x-4)}$, $x \in \mathbb{R}$, $x \neq 4$. Sketch the graph of *f*. *Hint*: Start with the graph of $\frac{1}{x}$ and perform transformations on it.

1.2 Coordinate Geometry

The equation of a Straight Line

We are familiar with straight lines of the form $y = mx + c$, where m is the gradient and c is the intercept on the y-axis. Illustrated in *fig. 1.22*.

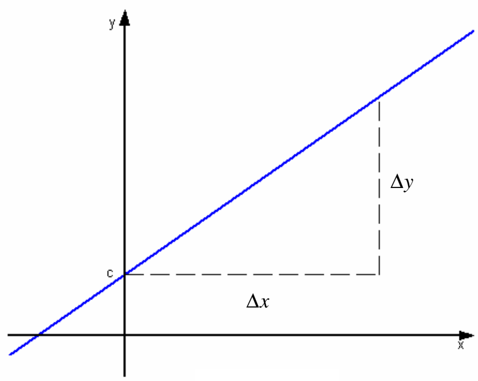


fig.1. 22

$$\text{gradient} = \frac{\Delta y}{\Delta x} = m$$

intercept on y-axis $= c$.

We can work out the equation of a straight line given any two points on the line, or the gradient of the line and one point on the line.

Example 1.26 Find the equation of the straight line which passes through the points $(-2,-12)$ and $(4,6)$.

We have the two points $(x, y) = (-2,-12)$ and $(x, y) = (4,6)$. We can find the gradient from this information, because $\text{gradient} = \dfrac{\Delta y}{\Delta x} = \dfrac{18}{6} = 3$, as illustrated in *fig.1. 23*.

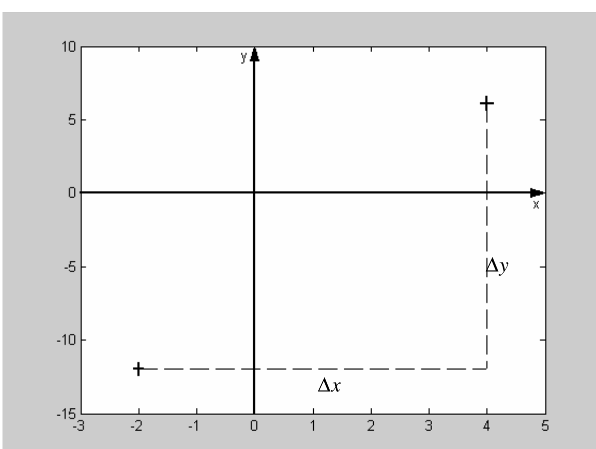


fig. 1.23

So the equation has the form $y = 3x + c$. To find c, we substitute either one of the given points into the equation $y = 3x + c$. Let us substitute in the point $(4,6)$. This gives us that $6 = 3\times 4 + c$ $\Rightarrow c = -6$. So the required equation is $y = 3x - 6$

Test 1.14 Find the equation of the straight line with gradient -2 and which passes through the point $(1,-6)$.

Sometimes, the equation of a straight line may be given in the form $Ax+By+C=0$. In this case, to find the gradient and intercept, it is usually easiest to rearrange the equation into the standard form $y=mx+c$.

Example 1.26 Find the gradient and y intercept of the straight line $4y+12x-40=0$.

We can rearrange this into standard form as $y=-3x+10$. So the gradient is -3 and the y intercept is 10.

Test 1.15 Find the gradient of the straight line $4y+12x-40=0$. Also find where this line crosses both axes.

Test 1.16 Find the gradient of the straight line $\frac{y}{4}-\frac{x}{2}=3$. Also find where this line crosses both axes.

Distance Between Two Points

What is the distance between the points $(2,4)$ and $(8,12)$? The points are plotted in *fig. 1.24*.

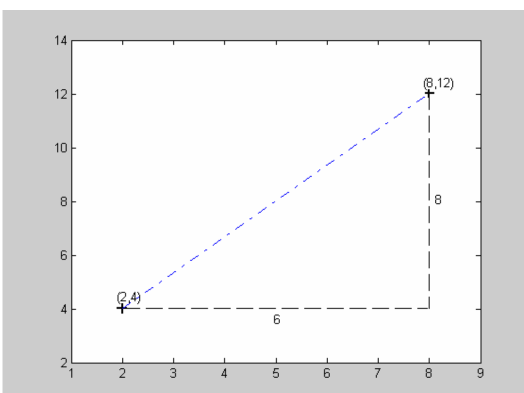


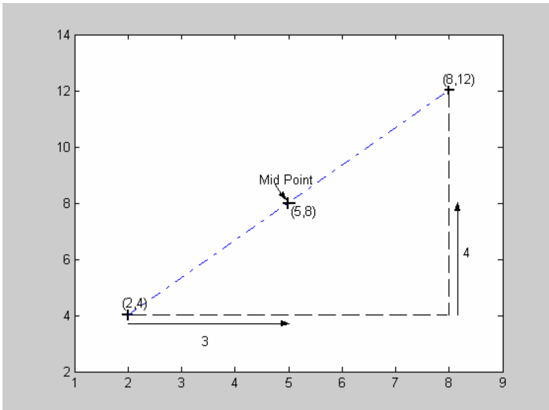
fig. 1.24

The horizontal distance between the two points in 6 units. The vertical distance between the two points is 8 units. From Pythagoras' Theorem, the (shortest) distance between the two points is $\sqrt{6^2+8^2}=10$ units.

The distance between points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Mid Point

Look back at *fig. 1.24*. What is the midpoint between $(2,4)$ and $(8,12)$? The midpoint lies on the straight line joining the two points and is equidistant to the two points. Look at the triangle in *fig.1. 25*. To find the midpoint, we imagine a vertical line bisecting the base of the triangle in half. Imagine a horizontal line bisecting the height of the triangle in half. Where these two lines meet is the midpoint of $(2,4)$ and $(8,12)$.



The coordinate of the midpoint of the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

fig. 1.25

Example 1.26 Find the distance between the points $(-1,3)$ and $(6,27)$.

Distance $=\sqrt{(-1-6)^2+(3-27)^2}=\sqrt{625}=25$.

Test 1.17 Find the distance between the points $A=(-3,-4)$ and $B=(-15,12)$. Find the coordinates of the midpoint of the straight line joining *A* and *B*.

Test 1.18 Prove that *ABC* is a right-angled triangle where $A=(2,1)$, $B=(5,-1)$, $C=(9,5)$.

Product of Gradients of Two Perpendicular Lines

When two straight lines are perpendicular, the product of their gradients is -1.

Look at *fig. 1.26*. This shows two perpendicular lines, l_1 and l_2. Let the angle that l_2 makes with the x-axis be θ. The angle that l_1 makes with the y-axis is also θ, as shown.

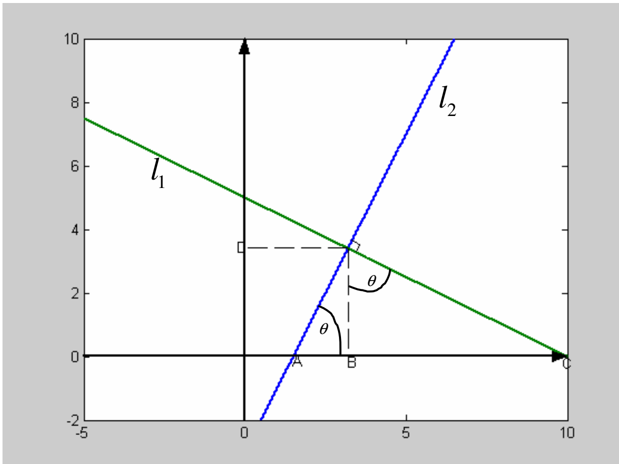


fig. 1.26

Let the gradient of l_2 be m_2. From the definition of the gradient, we have that $m_2 = \tan\theta$. We also have that $\tan\theta = \dfrac{BC}{BD}$. Hence, $m_2 = \tan\theta = \dfrac{BC}{BD}$. Now, $-\dfrac{BD}{BC}$ is the gradient of line l_1, let us call this m_1. So

$$m_2 = \tan\theta = \frac{BC}{BD} = -\frac{1}{m_1}.$$

Hence $m_2 = -\dfrac{1}{m_1}$. So the product of the gradients of two perpendicular straight lines is -1.

Example 1.27 Find the equation of the line which passes through the point $\left(-3, -\dfrac{3}{2}\right)$ and is perpendicular to the line $y = 2x + 4$.

If the line is to be perpendicular to the given line, it must have gradient $-\dfrac{1}{2}$, i.e. it must have the form $y = -\dfrac{1}{2}x + c$. Substituting in the given point which lies on this line allows us to find c. We have that,

$-\dfrac{3}{2} = -\dfrac{1}{2}\times(-3) + c \quad \Rightarrow \quad c = -3$. So the required equation is $y = -\dfrac{1}{2}x - 3$.

Test 1.18 Find the equation of the straight line which passes through the origin and is perpendicular to the line joining the points $(4,1)$ and $(1,4)$.

The equation of a Circle

Look at the circle in *fig. 1.27*. A radius has been drawn from the centre (origin) to a point on the circumference (x, y). What is the equation of this circle? We need to write down an equation involving *x* and *y* (and *r*). From Pythagoras' Theorem, we can write, $x^2 + y^2 = r^2$. This is the equation of a circle with centre at the origin and radius *r*. We can see that, in this case, $r = 2$, so the equation of the circle in *fig. 27* is $x^2 + y^2 = 2^2$, or $x^2 + y^2 = 4$.

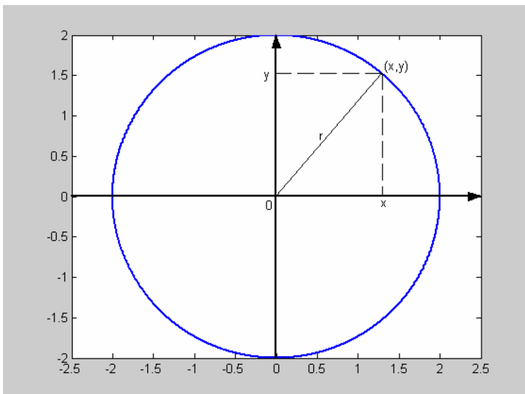


fig .1. 27

The equation of a circle with centre at the origin and radius *r* is $x^2 + y^2 = r^2$

What about a circle whose centre is not at the origin?

Study *fig. 1.28*. Here we have a circle with centre (*a*,*b*) and radius *r*. We can use Pythagoras' Theorem here to write down a similar expression for the right angled triangle (in blue), but this time, the base of the triangle has length *x* – *a* units, and the height of the triangle is *y* – *b* units. We have, $(x-a)^2 + (y-b)^2 = r^2$.

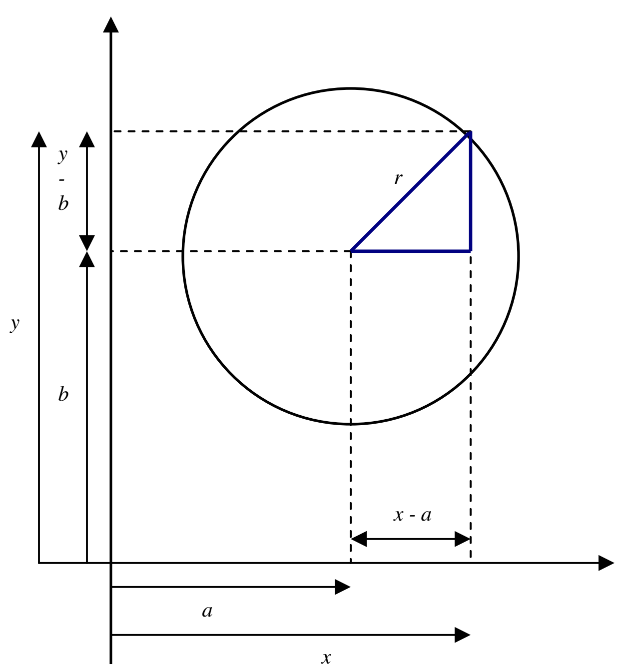


fig. 1.28

The equation of a circle with centre (*a*,*b*) and radius *r* is $(x-a)^2 + (y-b)^2 = r^2$

A circle with centre not at the origin can also be thought of as a translation of a circle with centre at the origin.

Example 1.28 Find the centre and radius of the following circle:

$$x^2 + 2x + y^2 - 6y - 6 = 0.$$

Here we have to rewrite the equation above in the standard form, $(x-a)^2+(y-b)^2=r^2$, so that we can read off the required information.

To do this, we need to complete the square.

$$x^2+2x+y^2-6y-6=0$$

$$\Rightarrow \quad (x+1)^2-1+(y-3)^2-9-6=0 \quad \Rightarrow \quad (x+1)^2+(y-3)^2=16$$

Now we can see that this is the equation of a circle with centre $(-1,3)$ and radius $\sqrt{16}=4$.

Example 1.29 Find the equation of the circle with centre at $(1,-2)$ and which passes through the point $(1,0)$.

Since we are told that the centre is at $(1,-2)$, we can immediately write:

$$(x-1)^2+(y+2)^2=r^2.$$

Now all that remains is to find *r*. To do this, we use the other piece of information given, i.e. that the circle passes through the point $(1,0)$, i.e. when $x=1,\ y=0$. So, setting $x=1$ and $y=0$ we have:

$$(0+2)^2=r^2 \quad \Rightarrow \quad r^2=4 \quad \Rightarrow \quad r=2$$

Notice, we select the positive sign (we can't have a circle with a negative radius).

So the required equation is:

$$(x-1)^2+(y+2)^2=4.$$

Test 1.19 Find the equation of the circle with centre $(-1,-2)$ and radius 4.

Test 1.20 Find the centre and radius of the circle whose equation is $x^2+y^2+8x-2y-8=0$.

Circle Properties

- The angle in a semicircle is a right angle.
- The perpendicular from the centre to a chord bisects the chord
- The tangent to a circle is perpendicular to the radius at its point of contact

Task: Draw pictures to illustrate the above circle properties.

Tangents and Normals

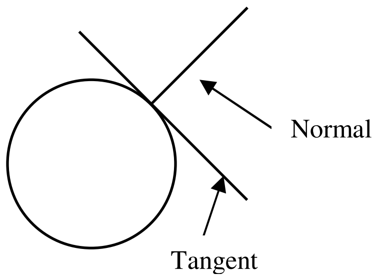


fig. 1.29

A line that just touches a circle (or curve) but does not actually cross it is called a **tangent**. A line that is at right angles to another line is said to be **normal** to it, as illustrated in *fig. 1.29*.

Example 1.30 Find the equation of the tangent to the circle $x^2 + y^2 + 6x - 4y + 8 = 0$ at the point $(-1,1)$.

Our first job is to write the equation of the circle in standard form. Completing the square gives the equation as $(x+3)^2 + (y-2)^2 = 5$ (check). Now let us make a sketch of the circle and the point $(-1,1)$, see *fig. 1.30*.

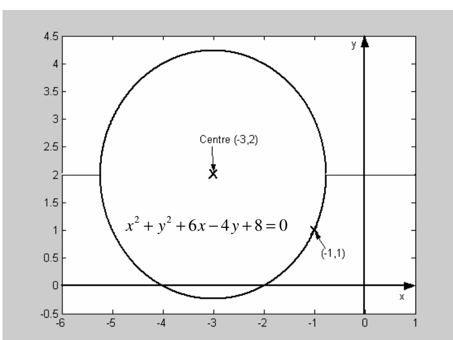


fig. 1.30

In order to find the equation of the tangent, we must first find the equation of the radius joining the points $(-3,2)$ and $(-1,1)$. We then use one of the circle properties, namely that the tangent to a circle is perpendicular to the radius at its point of contact, plus the point $(-1,1)$, which lies on the tangent, to find the equation of the tangent.
First we find the equation of the radius joining the points $(-3,2)$ and $(-1,1)$. We can easily see that this line has gradient $-\frac{1}{2}$, so it has the form $y = -\frac{1}{2}x + c$. To find *c*, we use one on the given points that lies on the line. We will use the point $(-1,1)$ (we could have used the point $(-3,2)$). Substituting in $(-1,1)$ gives us that $1 = \frac{1}{2} + c \quad \Rightarrow \quad c = \frac{1}{2}$. So the equation of the radius joining the points $(-3,2)$ and $(-1,1)$ is $y = -\frac{1}{2}x + \frac{1}{2}$.

Now the equation of the tangent is perpendicular to this line at the point of contact, $(-1,1)$, so it has gradient 2 (product of the gradients is -1). So the equation of the tangent has the form $y=2x+c$. To find c we use the given point that lies on the line, namely $(-1,1)$. Substituting this in gives $1=-2+c \quad \Rightarrow \quad c=3$. So the equation of the tangent at the point $(-1,1)$ is $y=2x+3$.

Test 1.21 Show that the point $(6,3)$ lies on the circle $x^2+y^2-10x-12y+51=0$. Find the equation of the tangent to the circle at the point $(6,3)$.

Test 1.22 The point $(9,p)$ lies on the circle $x^2+y^2-14x+8y+57=0$. Find p. Find the equation of the normal of the tangent at the point $(9,p)$. Find the equation of the tangent to the circle at the point $(9,p)$.

1.3 Differentiation

Introduction

We are familiar with finding the gradients of straight lines. The gradient of a straight line is a measure of how steep the line is. Does it make any sense to talk about the gradient of a curve? Look back at *fig. 1.2*, which shows the plot of $f(x)=x^2$. What is the gradient of $f(x)=x^2$? Unlike a straight line, which has a constant steepness, $f(x)=x^2$ is a curve and so has no fixed steepness. The graph is steeper at the point $x=4$ than it is at the point $x=1$. At the point $x=0$, the graph is 'flat'; it has no steepness here. When working with curves, we cannot give a constant gradient in the same way that we can for straight lines. Instead, we have to talk about the gradient of a curve *at a particular point*. So how do we find the gradient of a curve at a particular point? Let us return to the graph of $f(x)=x^2$, which is drawn again in *fig. 1.31*. What is the gradient of this curve at the point $x=3$? One way to think about this is to draw a **tangent** to the curve at the point $x=3$. This tangent is, of course, a straight line and so we can find its gradient.

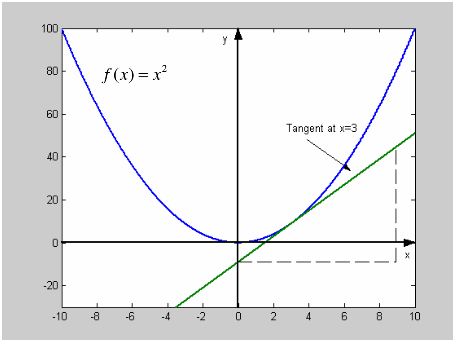


fig. 1.31

In fact we define the gradient of the curve $f(x)=x^2$ at the point $x=3$ to be the gradient of the tangent of the curve at the point $x=3$. If we were to draw a tangent to the curve at the point $x=8$, the tangent would be steeper, and so the gradient of the curve at the point $x=8$ would be a higher value than the gradient of the curve at the point $x=3$. The gradient of the curve at the point $x=0$ is a horizontal line, so the gradient of the curve at $x=0$ is zero.

The Gradient of the Tangent as a Limit

fig. 1.32 shows an arbitrary function $f(x)$. We want to find the gradient of $f(x)$ at the point *P*, where $x=a$. We call the gradient of $f(x)$ at the point $x=a$ the **derivative** of $f(x)$ at $x=a$. The derivative of $f(x)$ at $x=a$ can be written as $f'(a)$, or

$$\frac{\mathrm{d}f(a)}{\mathrm{d}x} \quad \text{or} \quad \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=a}.$$

As we have seen, the gradient of $f(x)$ at the point $x=a$ is the gradient of the tangent of $f(x)$ at $x=a$. Drawing in a tangent by hand and measuring the gradient is a time consuming and inaccurate way to proceed. Instead we consider another point on $f(x)$, point *Q*, and draw in the line joining *P*and *Q*. This line is called a **chord**. The point *Q* has *x*-coordinate, $x=a+h$.

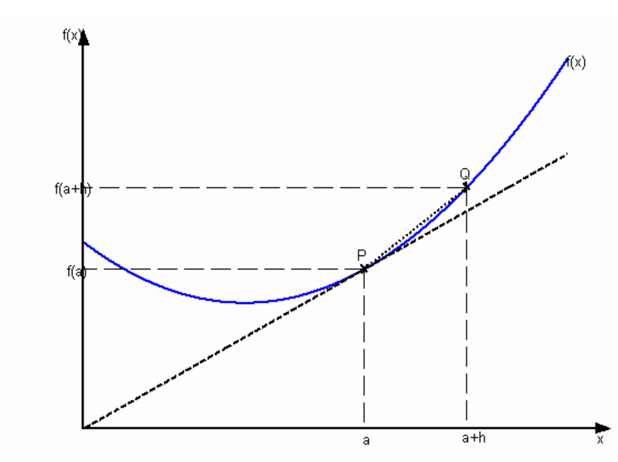


fig. 1.32

Now we can see from the diagram that the gradient of the chord *PQ* is approximately equal to the gradient of the tangent at *P*. We can see from the diagram that the gradient of the chord *PQ* is $\dfrac{f(a+h)-f(a)}{h}$

Imagine that the point *Q* slides down $f(x)$ so that it is closer to *P*, i.e. *h* decreases.

The closer Q gets to P, the closer the gradient of the chord PQ gets to the gradient of the tangent at P. We say that *'in the limit as h tends to zero'*, the gradient of the chord PQ equals the gradient of the tangent at P. This means that as h decreases and Q gets closer to P, the gradient of the chord PQ gets closer to the gradient of the tangent at P. As we make h very small, so that Q and P are very close, the gradient of the chord PQ gets very close to the gradient of the tangent at P. We can make h as small as we like. As h is made arbitrarily small, so Q and P become arbitrarily close together, the gradient of PQ becomes arbitrarily close to the gradient of the tangent at P. We write:

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right).$$

Another notation is often used for the derivative. If $y = f(x)$, then the notation $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ is often used. Note $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not a fraction, it is just a piece of notation to stand for the derivative of y with respect to x ('with respect to x' just means that the variable is x; if we had $y = f(t)$, then the derivative would be written $\frac{\mathrm{d}y}{\mathrm{d}t}$).

Let us look at a particular example. Let us consider the simplest quadratic equation $y = x^2$. Suppose we want to find the gradient of the graph of $y = x^2$ at a general point x.

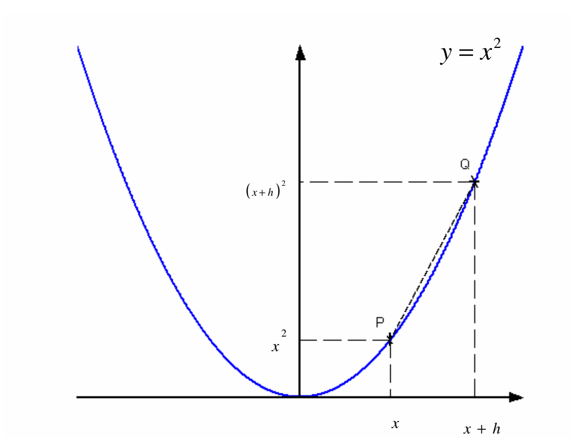


fig. 1.33

The gradient of the chord PQ is:

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h} = 2x + h$$

The gradient at the general point P is the limit of the gradient of the cord as $h \to 0$. This means that we make h smaller and smaller (approach zero).

We make h so small that it becomes insignificant, so $2x+h$ becomes $2x$ as h tends to zero. Therefore, the gradient of the graph of $y = x^2$ at any point x, is $2x$. We write:

$$f'(x) = \lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{(x+h)-x}\right) = \lim_{h\to 0}(2x+h) = 2x, \text{ or}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h \to 0}\left(\frac{(x+h)^2 - x^2}{(x+h) - x}\right) = \lim_{h \to 0}(2x+h) = 2x.$$

For example, the gradient of the tangent of the graph of $y = x^2$ at the point $x = 4$ is $2x = 2 \times 4 = 8$.

A similar argument can be make to find the derivative of $y = x^3$. It turns out that the derivative of $y = x^3$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ We cal also use this argument to find the derivative of $y = x^4$, although this is more tricky. If you look back at the argument for $y = x^2$, you will see that to find the derivative of $y = x^4$ from first principles, we will have to expand $(x+h)^4$, which takes some time (unless you know a shortcut). It turns out, however, that the derivative of $y = x^4$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$. From these few examples, can you work out what the derivative of $y = x^5$ is? The answer is $\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4$.

The Derivative of $y = ax^n$

The derivative of $y = x^n$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$. More generally the derivative of $y = ax^n$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = nax^{n-1}$ (where *a* and *n* are constants).

If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nax^{n-1}$

This result is valid for all $a,\ n \in \mathbb{R}$.

Example 1.31 Find the gradient of the tangent to the curve $y = 3x^5$ at the point $x = 2$.

Following the rule we have:

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times 5x^{5-1} = 15x^4$. So the gradient of the tangent at $x = 2$ is:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = 15 \times 2^4 = 240.$$

The notation $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2}$ stands for 'the derivative of *y* with respect to the variable *x*, evaluated at $x = 2$'.

Example 1.32 Find the coordinates of the point *P* which lies on the curve $f(x) = 2\sqrt{x}$ such that the tangent to $f(x)$ at *P* has gradient $\frac{1}{3}$.

First we write $f(x) = 2\sqrt{x}$ as $f(x) = 2x^{\frac{1}{2}}$. Then we differentiate,

$$f'(x) = 2\times\frac{1}{2}x^{\frac{1}{2}-1} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}.$$

Now we want the gradient of the tangent to be $\frac{1}{3}$, i.e. we want $f'(x) = \frac{1}{\sqrt{x}} = \frac{1}{3}$, so we must have that $x = 9$, i.e. $f'(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$.

When $x = 9$, $f(9) = 2\sqrt{9} = 6$. So the coordinates of the point *P* are $(9, 6)$.

Test 1.23 Given that $g(m) = \frac{-2}{m^3}$, find $g'(81)$. *Hint*: First write $g(m)$ as $g(m) = -2m^{-3}$ and then follow the rule.

Sum / Difference of Functions

If we have an expression which consists of a sum or difference of several terms, for example, $y = 2x^2 - x^4$, then we differentiate the expression by differentiating each term individually, i.e. $y' = 2\times 2x^{2-1} - 1\times 4x^{4-1} = 4x - 4x^3 = 4x(1 - x^2)$.

If $y = f_1(x) + f_2(x) + \ldots + f_n(x)$ then $y' = f_1'(x) + f_2'(x) + \ldots + f_n'(x)$

Derivative of a Constant

If a function simply consists of a constant term, $y = a$, where *a* is a constant, then $y' = 0$. We can see why this is true, because $y = a$ can be written as $y = ax^0$ (remember, anything to the power zero is one) and following the rule we have $y' = 0\times ax^{0-1} = 0$. Also, the graph of $y = a$, where a is a constant, is a straight, horizontal line, and so has zero gradient.

Example 1.33 If $p = \frac{4}{t} + 6$, find $\frac{\mathrm{d}p}{\mathrm{d}t}$.

First we write, $p = 4t^{-1} + 6$. Following the rules, remembering that the derivative of a constant is zero, we have:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -1 \times 4t^{-1-1} + 0 = -4t^{-2} = -\frac{4}{t^2}.$$

We see that adding any constant to a function does not change its derivative. Look back at the section on graphical transformations. We saw that the transformation $f(x) \rightarrow f(x) + a$ is a translation by a units parallel to the y-axis, i.e. adding a constant to a function simply translates it, it does not change its gradient at any point.

Test 1.24 Given that $g(d) = 3d^2 + \frac{1}{d} - \sqrt{d} - \pi$ find $g'(d)$.

Test 1.25 The function $y = 3x^n + x$, where n is an integer, has a tangent at the point $x = 1$ with gradient -8. Find n.

Interpretation as a Rate of Change

Suppose that y is a linear function of x. The gradient of the line tells us the rate at with y changes with respect to x. For example, the line $y = 2x$ has gradient 2. This means that for every 2 units the line moves up in the y direction, the line moves 1 unit along in the x direction.

The simplest way to consider a function as a rate of change is to consider a velocity function. If a body is moving with constant velocity, then the velocity can be calculated by dividing the distance traveled by the time taken. Alternatively, if we plot distance traveled against time, then the gradient of the line will give the velocity. *fig. 1.34* shows the distance against time plots for two bodies A and B.

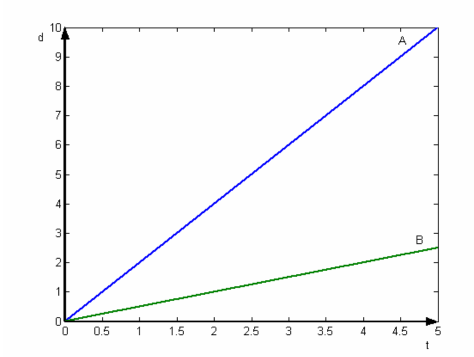


fig. 1.34

Which body is travelling faster? The gradient of the lines represents the velocity (distance divided by time). The higher the gradient, the higher the velocity, i.e. the higher the rate of change of distance with respect to time. Therefore, body *A* has a greater velocity than body *B*. We can see that the gradient of line *A* is 2, while the gradient of line *B* is $\frac{1}{2}$. Body *A* therefore has a velocity of 2 ms^{-1}, whilst body *B* has a velocity of $\frac{1}{2} \text{ ms}^{-1}$ (Assuming that distance is measured in metres and time is measured in seconds).

Now consider *fig. 1.35*. This shows the distance – time plot for body *C*.

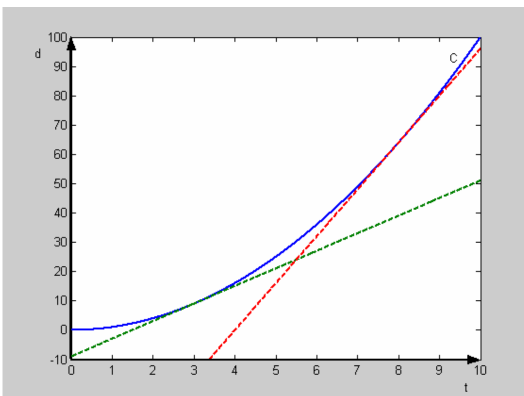


fig. 1.35

What is the velocity of body *C*? This question is essentially asking what is the gradient of line *C*. As we now know, line *C* does not have a fixed gradient. Body *C* does not have a fixed velocity, i.e. it is not travelling at constant velocity. The gradient of the line represents the velocity, but the gradient of *C* is different at different points. Tangents to *C* have been drawn at times $t = 3$ and $t = 8$. Of course, body *C* has a greater velocity at $t = 8$ than it does at $t = 3$, because the gradient of the tangent is steeper here. The tangent at $t = 3$ has gradient 6. This means that when $t = 3$, the rate of change of distance with respect to time is 6 ms^{-1}. The tangent at $t = 8$ has gradient 16. This means that when $t = 8$, the rate of change of distance with respect to time is 16 ms^{-1}. The gradients of the tangents of line *C* are continuously increasing. The gradient of the tangent at $t = 0$ is zero, and the gradients of the tangents increase as *t* increases. Body *C* as *accelerating*.

Stationary Points

fig. 1.36 shows a plot of $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 18x + 7$. There are two special points on this curve where the gradient of the tangent is zero. These points are $x = -6$ and $x = 3$.

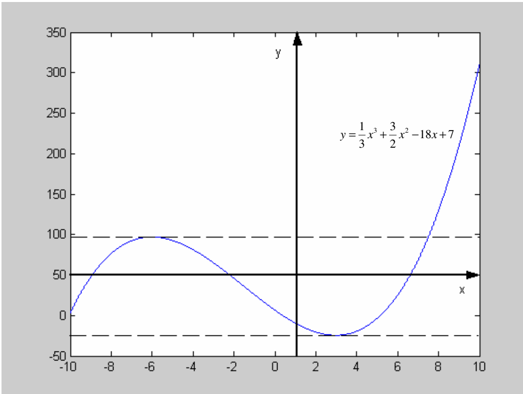


fig. 1.36

The tangents at these two points have been drawn on the graph. The points where the tangent of the curve has gradient zero are called **stationary points.** How do we find where the stationary points are on a curve? Let us use the example $y=\frac{1}{3}x^3+\frac{3}{2}x^2-18x+7$. The derivative of this function is $y'=x^2+3x-18$. Now the stationary points occur where the derivative is zero, i.e. where $x^2+3x-18=0$. Factorising and solving this gives, $(x+6)(x-3)=0 \Rightarrow x=-6$ or $x=3$. These are the stationary points. Notice, the turning point at $x=-6$ corresponds to a **local maximum**, whereas the turning point at $x=3$ corresponds to a **local minimum**.

Example 1.34 Find the coordinates of the stationary point of the function $y=(x-3)(2x+4)$.

We need to differentiate *y*. First, we need to multiply the brackets out:

$y=2x^2-2x-12 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=4x-2$. Stationary points occur when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, i.e. when:

$4x-2=0 \quad \Rightarrow \quad x=\frac{1}{2}$. When $x=\frac{1}{2}$, $y=2\times\left(\frac{1}{2}\right)^2-2\times\frac{1}{2}-12=-12\frac{1}{2}$.

So the coordinates of the stationary point are $\left(\frac{1}{2},\ 12\frac{1}{2}\right)$.

Test 1.26 Find the *x*-coordinates of the stationary points of the curve $y=\frac{1}{3}x^3+2x^2-32x$. Does each stationary point correspond to a local maximum or a local minimum?

Test 1.27 Find the turning points of the function $y = x^2 - 49$. Make a sketch of the function, marking on where the function crosses the *x*-axis and the coordinates of the stationary point.

Increasing and Decreasing Functions

A function that has a positive gradient everywhere is called a (strictly) **increasing function.** A function that has a negative gradient everywhere is called a (strictly) **decreasing function.** *fig. 1.37* shows an example of an increasing function, $f(x)$. *fig. 1.38* shows an example of an decreasing function, $g(x)$.

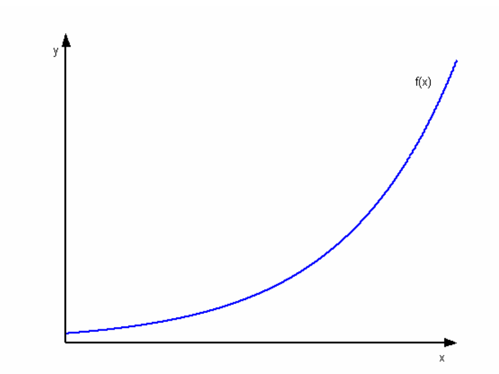


fig. 1.37

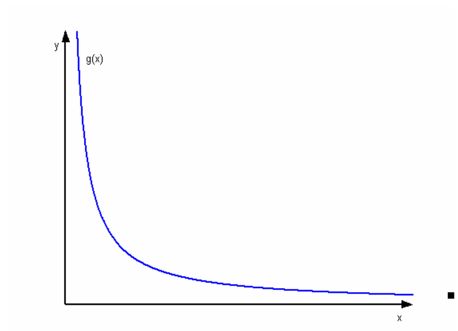


fig. 1.38

If a function has a positive gradient at a particular point, we say that the function is increasing at that point. If a function has a negative gradient at a particular point, we say that the function is decreasing at that point.

Example 1.35 Is the function $y = 3x^3 - 7x^2 + 2x - 9$ increasing or decreasing at the points $x = 1$, $x = 2$? What can you say about the graph between the points $x = 1$ and $x = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 - 14x + 2$$

$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 9 \times 1^2 - 14 \times 1 + 2 = -3$, so the function is decreasing at $x = 1$.

$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = 9 \times 2^2 - 14 \times 2 + 2 = 6$, so the function is increasing at $x = 2$.

If the function is decreasing at $x = 1$ and increasing at $x = 2$, then there must be (at least one) turning point (at least a local minimum) between these two points, as illustrated in *fig. 1.39*.

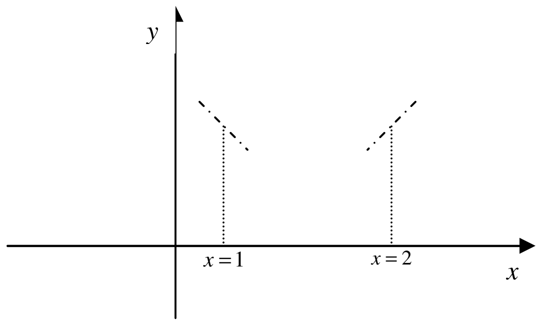


fig. 1.39

Test 1.28 Consider the function $g(t) = \frac{4}{3}x^3 - 4x + 5$. Is this function increasing, decreasing or stationary at the points $x = 0$, $x = 1$ and $x = 2$? Is the stationary point a local maximum or a local minimum?

Second Order Derivatives

We can differentiate a given function more than once. Consider $y = 2x^3 - x^2 + 3x - 4$. Differentiating this function once gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2x + 3$. We can differentiate the function a second time, the symbol we use to denote the second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, we have that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x - 2$. Notice, as with $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is just a symbol, a piece of notation. It is not a fraction and, despite the appearance of '2', nothing is squared! It is simply a symbol to say that we have differentiated twice.

When the function notation is used, for example given a function $f(x)$, the symbol for the derivative is $f'(x)$ and the symbol for the second derivative is $f''(x)$.

Example 1.36 Given $y = (2x + 3)^3$, find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

First we expand the bracket:

$$y=(2x+3)^3=(2x+3)(4x^2+6x+9)$$

$$=8x^3+12x^2+18x+12x^2+18x+27$$

$$=8x^3+24x^2+36x+27.$$

Now, differentiation once gives:

$$\frac{dy}{dx}=24x^2+48x+36.$$

Differentiating a second time gives:

$$\frac{d^2y}{dx^2}=48x+48=48(x+1)$$

Test 1.29 Given that $f(x)=x(2x-1)^3$, find $f''(x)$

Test 1.30 Given that $y=(2x-7)^2$, find $\frac{d^2y}{dx^2}$

Applications to Determining Maxima and Minima

If a function $d=f(t)$ represents the relationship between the distance traveled and time taken of a body, then we have seen that the derivative represents the rate of change of distance with respect to time, the velocity. The second derivative represents the rate of change of the rate of change of distance with respect to time, or the rate of change of velocity with respect to time, which is the acceleration.

We can use the second derivative to determine whether a stationary point is a local maximum or local minimum. If a function $f(x)$ has a stationary point at $x=a$, i.e. $f'(a)=0$, to determine whether this is a maximum or a minimum we compute $f''(a)$. If $f''(a)>0$, then the turning point is a local minimum. If $f''(a)<0$, then the turning point is a local maximum.

If a function $y=f(x)$ has a stationary point at *P*, then if:

$\frac{d^2y}{dx^2}>0$ at *P*, the stationary point is a local minimum

$\frac{d^2y}{dx^2}<0$ at *P*, the stationary point is a local maximum.

! Remember: $\frac{d^2y}{dx^2} > 0$ corresponds to a **minimum**, while $\frac{d^2y}{dx^2} < 0$ corresponds to a **maximum**.

Example 1.37 Find the stationary points of the function $f(x) = x^3 - 15x + 2$. Determine whether these stationary points are maxima or minima.

Differentiation gives, $f'(x) = 3x^2 - 15$. Stationary points occur when

$f'(x) = 3x^2 - 15 = 0$, i.e. when $x^2 = \frac{15}{3} \Rightarrow x = \pm\sqrt{5}$.

To determine whether these stationary points are maxima or minima, we need to compute the second derivative, $f''(x) = 6x$.

Now we have that, $f''(\sqrt{5}) = 6\sqrt{5}$, which is positive, so at $x = \sqrt{5}$ we have a local minimum.

$f''(-\sqrt{5}) = -6\sqrt{5}$, which is negative, so at $x = -\sqrt{5}$ we have a local maximum.

Test 1.30 Find the stationary points of the curve $f(x) = \frac{4}{3}x^3 - 15x^2 + 14x - 10$. Using the second derivative test, determine whether these stationary points are maxima or minima.

Optimisation Problems

Example 1.38 A soft drinks manufacturer is designing new packaging for its fizzy drink. The container for the drink will be a can made from thin aluminium and is to have a capacity of $333 \text{ ml} = 333 \text{ cm}^3$. Varying the height and radius of the container will vary the amount of aluminium that is needed for each can. What should the height and radius of the can be so that the minimum amount of aluminium is needed?

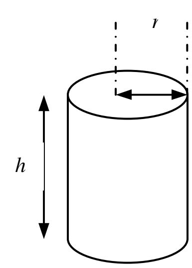


fig. 1.40

Now, the surface area of the can is given by:

$S = 2\pi r^2 + 2\pi rh$ ....**(1)**

The problem with this equation is that is has two variables, *r* and *h*. We need to eliminate one of them.

To do this we can use the equation for the volume of the can, which is given by:

$V = \pi r^2 h$ ....(**2**)

We know that the volume of the can is to be 333 cm^3. We can substitute this into equation (**2**) and rearrange for *h*. We can then use this to eliminate *h* from equation (**1**). From (**2**) we have:

$$333 = \pi r^2 h \quad \Rightarrow \quad h = \frac{333}{\pi r^2}$$

Substituting for *h* in (**1**) gives:

$$S = 2\pi r^2 + 2\pi r \times \frac{333}{\pi r^2} \quad \Rightarrow \quad S = 2\pi r^2 + \frac{666}{r}$$

The minimum / maximum surface area occurs when $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$.

Now $S = 2\pi r^2 + 666 r^{-1}$, so $\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - 666 r^{-2} = 4\pi r - \frac{666}{r^2}$.

So, $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$ when $4\pi r - \frac{666}{r^2} = 0$. Multiplying throughout by r^2 gives:

$$4\pi r^3 - 666 = 0 \quad \Rightarrow \quad r = \sqrt[3]{\frac{666}{4\pi}} = 3.76 \text{ cm} \text{ to three significant figures.}$$

Now we use the rearranged form of equation (**2**) to find the corresponding value of *h*:

$h = \frac{333}{\pi r^2}$, so when $r = 3.76$, $h = \frac{333}{\pi \times 3.76^2} = 7.50 \text{ cm}$ to three significant figures.

We need to make sure that these optimized values correspond to minima and not maxima. Performing the second derivative test on $S = 2\pi r^2 + \frac{666}{r}$, for $r = 3.76$ reveals that the optimized values are indeed minimum (check).

Test 1.31 At a speed of *x* km/hour, a vehicle can cover *y* km on 1 litre of fuel, where:

$$y = 5 + \frac{x}{2} + \frac{x^2}{60} - \frac{x^3}{1800}.$$

Calculate the maximum distance which the vehicle can travel on 30 litres of fuel.

1.4 Integration

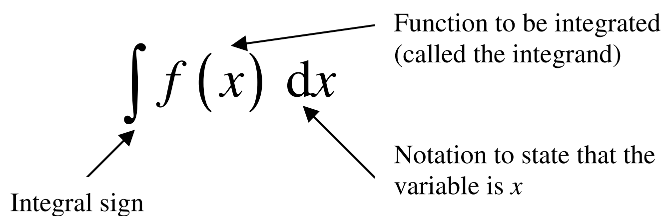
Introduction

Given that $f'(x) = 9x^2$ can we say what $f(x)$ is? Look back at the rule for differentiating $y = ax^n$. To differentiate an expression of this form, we multiply the coefficient by the exponent, and then subtract one form the exponent. To reverse this process, we simply add one on to the exponent and then divide the coefficient by the new exponent. So, following this rule, if $f'(x) = 9x^2$, then $f(x) = \frac{9}{2+1}x^{2+1} = 3x^3$. This

seems to be all well and good, if we differentiate our suggestion for $f(x)$, we do indeed get $f'(x) = 9x^2$. There is a slight problem, however. If we differentiate $f(x) = 3x^3 + 2$, we also get $f'(x) = 9x^2$. In fact, if we differentiate $f(x) = 3x^3 + c$, where *c* is any constant, we get $f'(x) = 9x^2$.

The Integral of $y = ax^n$

Integration can be thought of as the reverse process of differentiation. To integrate $y = ax^n$, we reverse the process for differentiation, i.e. we add one to the index and divide the coefficient by the new index. We must also add on a constant, *c*. This constant is called the **constant of integration**. This constant of integration is necessary for the reason discussed above; differentiating eliminates constant terms and so when reversing this process, we must be aware that constant terms may have been eliminated by the differentiation process. The notation used to stand for the integral of a function $f(x)$ with respect to a variable *x* is as follows:



The symbol $\int$ is the symbol for integration. The symbol $\mathrm{d}x$ simply means that we are integrating with respect to the variable *x*. If the variable were *t*, then this symbol would be replaced with $\mathrm{d}t$.

The integral of ax^n is $\frac{a}{n+1}x^{n+1} + c$, where *c* is the constant of integration. We write:

$$\int ax^n \; \mathrm{d}x = \frac{a}{n+1}x^{n+1} + c$$

! It is important not to miss out the constant of integration.

This result is valid for all $a,\ n \in \mathbb{R}$, except for $x = -1$.

If $x = -1$, then following this rule will give $\int ax^{-1} \; \mathrm{d}x = \frac{a}{-1+1}x^{-1+1} + c = \frac{a}{0} + c$, we are never allowed to divide by zero. This is a special integral which will be dealt with later.

In a similar way to differentiation, to integrate a sum or difference of terms, we simply integrate each term separately:

$$\int f_1(x) + f_2(x) + \ldots + f_n(x) \; \mathrm{d}x = \int f_1(x) \; \mathrm{d}x + \int f_2(x) \; \mathrm{d}x + \ldots + \int f_n(x) \; \mathrm{d}x$$

Note: A constant, a, can be written as ax^0, and so integrating this gives,

$\int ax^0 \; \mathrm{d}x = \frac{a}{0+1}x^{0+1} + c = ax + c$. For example $\int 2 \; \mathrm{d}x = 2x + c$.

Example 1.39 Calculate $\int 2x^2 + 8x^3 \; \mathrm{d}x$.

Following the rule, we have:

$$\int 2x^2 + 8x^3 \; \mathrm{d}x = \frac{2}{2+1}x^{2+1} + \frac{8}{3+1}x^{3+1} + c = \frac{2}{3}x^3 + 2x^4 + c.$$

Example 1.40 Calculate $\int \sqrt{x} - \frac{1}{x^2} \; \mathrm{d}x$.

First we rewrite $\sqrt{x} - \frac{1}{x^2}$ as $x^{\frac{1}{2}} - x^{-2}$. Then, following the rule we have:

$$\int x^{\frac{1}{2}} - x^{-2}\ \mathrm{d}x = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} - \frac{1}{-2+1}x^{-2+1} + c$$

$$= \frac{1}{3/2}x^{\frac{3}{2}} + x^{-1} + c$$

$$= \frac{2}{3}\left(\sqrt{x}\right)^3 + \frac{1}{x} + c$$

Test 1.32 Calculate $\int 15x^4 + 16x^3 - x + 7\ \mathrm{d}x$.

Test 1.33 Calculate $\int \frac{1}{x^4} - \frac{1}{x^9}\ \mathrm{d}x$.

Area Under a Curve

We know that the derivative of a function represents the gradient of the function (at a particular point). The integral of a function represents the area under the curve of the function. *fig. 1.41* shows part of a function $f(x)$. The shaded area, i.e. the area bounded by the curve the *x*-axis and the lines $x = a$ and $x = b$, is given by:

$$\int_a^b f(x)\ \mathrm{d}x = \int f(b)\ \mathrm{d}x - \int f(a)\ \mathrm{d}x$$

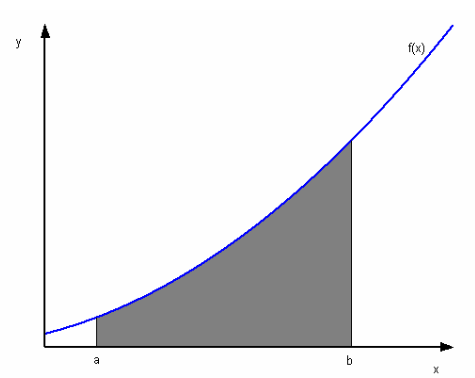


fig. 1.41

The symbol $\int_a^b$ means we are integrating between two limits, *a* and *b*. This method is best illustrated by an example.

Example 1.41 Calculate $\int_4^{10} \frac{1}{3}x^2 + 1\ \mathrm{d}x$.

First we integrate the function $\frac{1}{3}x^2 + 1$, then we substitute in the limits. This is usually denoted by putting square brackets around the integrated function, with the limits at the top and bottom of the bracket on the right hand side, as follows:

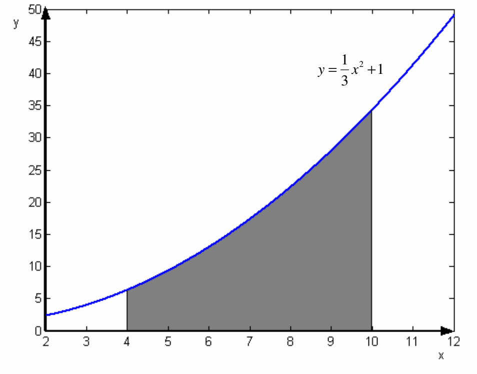


fig. 1.42

$$\int_4^{10} \frac{1}{3}x^2+1\,\mathrm{d}x=\left[\frac{1}{9}x^3+x\right]_4^{10}.$$

Next we substitute in the limits. We first substitute in $x=10$ and subtract from this the value of x^3+x when $x=4$, as follows:

$$\left[\frac{1}{9}x^3+x\right]_4^{10}=\left(\frac{10^3}{9}+10\right)-\left(\frac{4^3}{9}+4\right)$$

$$=\frac{1090}{9}-\frac{100}{9}=110.$$

So the result is:

$$\int_4^{10} \frac{1}{3}x^2+1\,\mathrm{d}x=110$$

This corresponds to the shaded area shown in *fig. 1.42*, which is 110 units^2.
This type of integral, where we integrate between two limits, is called a **definite integral**. An integral which does not involve limits, like example 1.39, is called an **indefinite integral**.

Note: It is not necessary to add a constant of integration when working with definite integrals, as this always cancels out. If we had added a constant of integration when calculating example 41, we would have got:

$$\int_4^{10} \frac{1}{3}x^2+1\,\mathrm{d}x=\left[\frac{1}{9}x^3+x+c\right]_4^{10}$$

$$=\left(\frac{10^3}{9}+10+c\right)-\left(\frac{4^3}{9}+4+c\right)$$

$$=\frac{1090}{9}+c-\frac{100}{9}-c=110.$$

The constant of integration always cancels out in this way for definite integrals.

Example 1.42 Calculate $\int_1^3 2x^3 + 4x^2 + 2 \, dx$. Illustrate the area under the graph of $y = 2x^3 + 4x^2 + 2$ that this integral represents.

$$\int_1^3 2x^3 + 4x^2 + 2 \, dx = \left[\frac{1}{2}x^4 + \frac{4}{3}x^3 + 2x \right]_1^3$$

$$= \left(\frac{1}{2} \times 3^4 + \frac{4}{3} \times 3^3 + 2 \times 3 \right) - \left(\frac{1}{2} \times 1^4 + \frac{4}{3} \times 1^3 + 2 \times 1 \right)$$

$= 78\frac{2}{3}$ (check). Thus, the shaded area in *fig. 1.43* is $78\frac{2}{3}$ units^2.

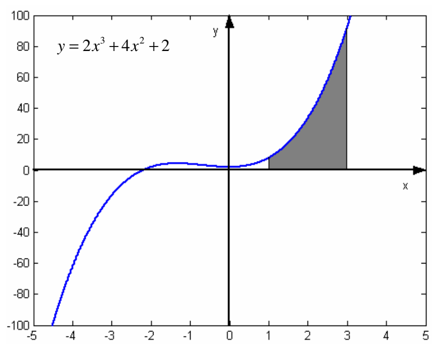


fig. 1.43

Test 1.34 Calculate the shaded area in *fig. 1.44.*

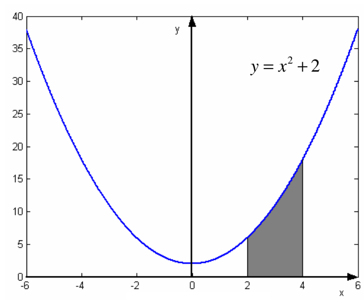


fig. 1.44

Test 1.35. Calculate $\int_1^2 8x^3 - 3x^2 + 4x + 6 \, dx$

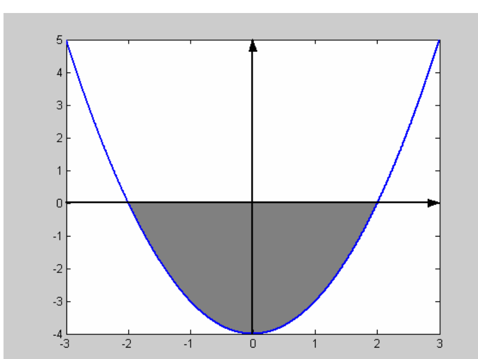


fig. 1.45

Example 1.43 Consider *fig. 1.45*. Let us find the shaded area, that is, the area bounded by the line $y = x^2 - 4$ and wholly below the x-axis. First, to find the limits of integration, we must calculate where the line $y = x^2 - 4$ cuts the x axis. The line $y = x^2 - 4$ cuts the x-axis when $y = 0$, i.e. when $x^2 - 4 = 0$. This occurs when $x = \pm\sqrt{4} = \pm 2$. So, the shaded area is given by:

$$\int_{-2}^{2} x^2 - 4 \, \mathrm{d}x = \left[\frac{1}{3}x^3 - 4x\right]_{-2}^{2}$$

$$= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)$$

$$= \frac{8-24}{3} - \left(\frac{-8+24}{3}\right)$$

$$= -\frac{32}{3} = -10\frac{2}{3}$$

Notice that the answer is negative. Whenever an integral represents an area that is wholly below the *x*-axis, the answer will be negative.

Test 1.36 *fig. 1.46* shows part of the graph of $y = (x+4)(x+1)(x-5)$. Find the area of the shaded region.

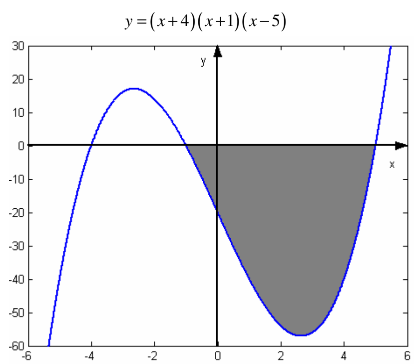


fig. 1.46

GCE AS/A Mathematics Chapter 2

2.1 Trigonometry

The Graphs of Sine and Cosine

Use your calculator to find the values of the functions $p(x)=\sin(x)$ and $q(x)=\cos(x)$ at 10° intervals in the range $0^\circ \le x \le 360^\circ$. Use these results to plot smooth graphs of the functions $p(x)=\sin(x)$ and $q(x)=\cos(x)$ in the interval $0^\circ \le x \le 360^\circ$. Your graphs should look like *fig. 2.1* and *fig. 2.2*.

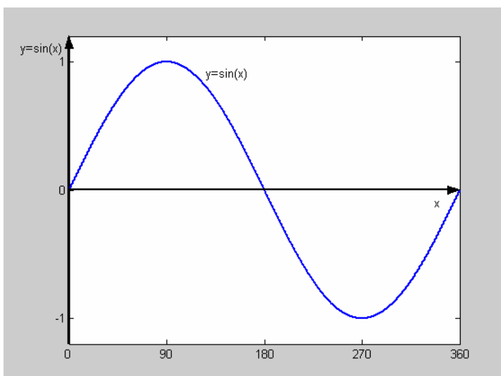


fig. 2.1

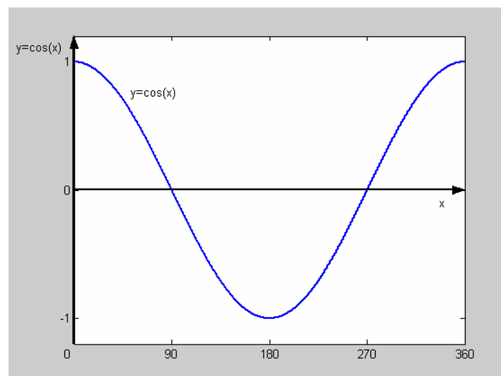


fig. 2.2

These graphs represent the basic shapes of the functions $\sin(x)$ and $\cos(x)$. We have plotted the functions in the range $0^\circ \le x \le 360^\circ$, but these functions are in fact valid for all $x \in \mathbb{R}$. However, we usually consider angles in the range 0 to 360 degrees, for example an angle of 370 degrees is in fact an angle of 10 degrees. The graphs of $\sin(x)$ and $\cos(x)$ are shown over a greater domain in *fig. 2.3* and *fig. 2.4*.

Notice that the period of the functions $\sin(x)$ and $\cos(x)$ is 360°. Notice that the graphs of $\sin(x)$ and $\cos(x)$ are quite similar. In fact, they have the same shape (sometimes called **sinusoidal**), they both have a minimum value of -1 and a maximum value of 1. Notice that the graph of $y=\cos(x)$ is symmetrical about the *y*-axis.

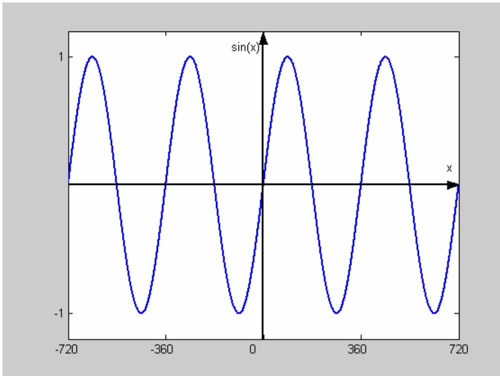


fig. 2.3

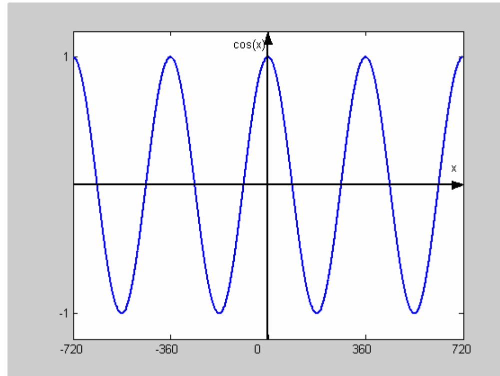


fig. 2.4

This gives rise to an interesting property of $\cos(x)$, that is:

$$\cos(x) \equiv \cos(-x).$$

Which can be seen directly from the graph. Functions that have the property $f(x) = f(-x)$ are called **even** functions.

The graph of $y = \sin(x)$ also has an interesting symmetrical property, that is:

$$\sin(-x) \equiv -\sin(x).$$

This can be seen directly from the graph. Functions that have the property $f(-x) = -f(x)$ are called **odd** functions.

Example 2.1 Given that $\sin 30^\circ = \frac{1}{2}$, without using a calculator, find $\sin(-30^\circ)$ and $\sin 210^\circ$.

With a little thought, and a good sketch of the graph, see *fig. 2.5*, we can see that,

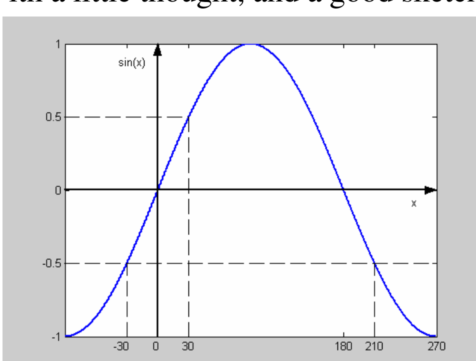


fig. 2.5

$$\sin(-30^\circ) = -\sin(30^\circ) = -\frac{1}{2}$$

and $\sin(210^\circ) = \sin(-30^\circ) = -\frac{1}{2}$.

Test 2.1 Given that $\cos\left(60^\circ\right)=\frac{1}{2}$, without using a calculator, find $\cos\left(240^\circ\right)$.

The Graph of Tangent

The function $f\left(x\right)=\tan\left(x\right)$ has a more unusually shaped graph. This is shown in *fig. 2.6.*

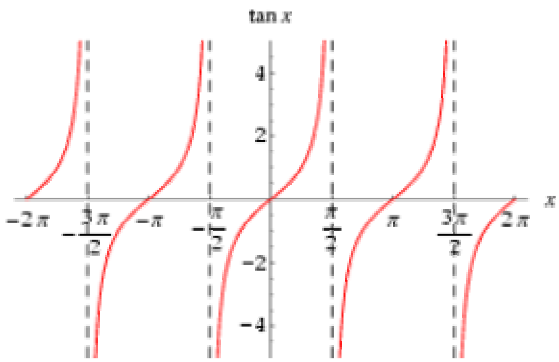


fig. 2.6
Note: radian measure used!

Note that, unlike $\sin\left(x\right)$ and $\cos\left(x\right)$, $\tan\left(x\right)$ is not bounded. For example, as $x\rightarrow 90^\circ$, $\tan\left(x\right)\rightarrow\infty$. Notice also that the graph of $f\left(x\right)=\tan\left(x\right)$ repeats itself every 180°.

The symmetry of the graph leads to the identity:

$$\tan\left(x\right)\equiv\left(x\pm 180^\circ\right)$$

Test 2.2 Given that $\tan 45^\circ=1$, use the symmetry of the graph of $f\left(x\right)=\tan\left(x\right)$ to find the value of $\tan 135^\circ$ without using a calculator.

Radian Measure

Up until now, we have measured angles in degrees, where one degree is $\frac{1}{360}$ of a full turn. From now on, we will almost exclusively use radian measure.

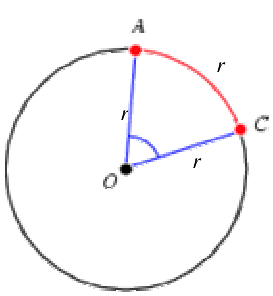


fig. 2.6

Consider a circle with an angle θ subtended by two radii. One **radian** corresponds to the angle which gives the same arc length as the radius, as shown in *fig. 2.6.*

If two radii subtend an angle θ, then the arc length, l, is given by $l=r\theta$, as shown in *fig. 2.7.*

We can see therefore, that the circumference of a circle divided by the radius of the circle will give the number of degrees in a circle, i.e. the number of degrees in a full turn.

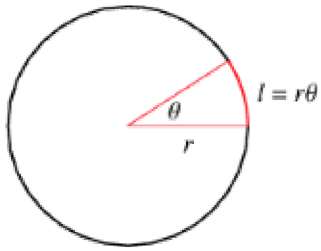


fig. 2.7

The number of degrees in a full circle is therefore given by:

$$\frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi r}{r} = 2\pi,$$

i.e. $360^\circ = 2\pi^c$, note a superscript 'c' is sometimes used to denote 'radian'.

$$180^\circ = \pi^c$$

Example 2.2 Convert **a)** 45° into radians **b)** 30° into radians.

a) Now, $180^\circ = \pi^c$, so $45^\circ = \dfrac{\pi^c}{4}$.

b) Now, $1^\circ = \dfrac{\pi^c}{180}$, so $30^\circ = \dfrac{30\pi^c}{180} = \dfrac{1}{6}\pi^c$.

Test 2.3 Convert 150° into radians.

! It will often be necessary to work with radians. When working with radians on your calculator, make sure it is switched to radian mode.

The Sine Rule

Look at *fig. 2.8*. There is a relation involving the sine of the angles at *A, B* and *C*, with the length of the sides *a*, *b* and *c*. The relation is as follows:

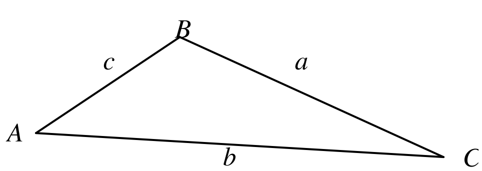


fig. 2.8

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and is called the **sine rule.**

A Proof of the Sine Rule

Look at *fig. 2.9*. From this diagram, we can see that,

$$\sin A = \frac{h}{c} \quad \Rightarrow \quad h = c\sin A \qquad \text{........................}\mathbf{(1)}$$

We can also see that,

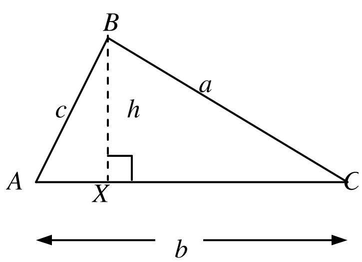


fig. 2.9

$$\sin C = \frac{h}{a} \quad \Rightarrow \quad h = a\sin C \text{ ....}\mathbf{(2)}$$

Since **(1)** = **(2)**, we can see that,

$$c\sin A = a\sin C \quad \Rightarrow \quad \frac{c}{\sin C} = \frac{a}{\sin A} \text{ ....}\mathbf{(3)}$$

By constructing a perpendicular form C to AB and giving a similar argument, we can deduce that,

$$a\sin B = b\sin A \quad \Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B}.$$

But we know from **(3)** that $\frac{a}{\sin A} = \frac{c}{\sin C}$, so,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. As required.
This is the sine rule. It relates the sides of any triangle to sine of its angles, whether the triangle is a right-angled triangle or not.

Example 2.3 Solve the triangle in *fig. 2.10*. (Solve the triangle means find all the missing sides and angles)

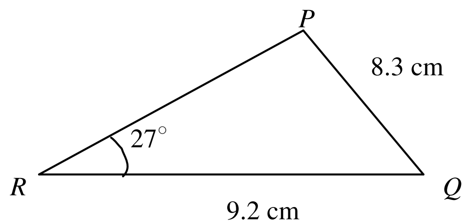


fig. 2.10

$$\frac{8.3}{\sin 27}=\frac{9.2}{\sin P} \quad\Rightarrow\quad \sin P=\frac{9.2}{8.3}\times\sin 27$$

$\Rightarrow\quad P=\sin^{-1}0.503=30.2^\circ$.

Now, $Q=180-(27+30.2)=122.8^\circ$.

So,

$$\frac{8.3}{\sin 27}=\frac{PR}{\sin 122.8} \quad\Rightarrow\quad PR=\frac{8.3}{\sin 27}\times\sin 122.8$$

$\Rightarrow\quad PR=15.4$ cm.

Example 2.4 Solve the triangle *ABC,* where $AC=30$ cm, $BC=20$ cm and $\angle CAB=30^\circ$.

First make a rough sketch and mark on the given information, shown in *fig. 2.11*.

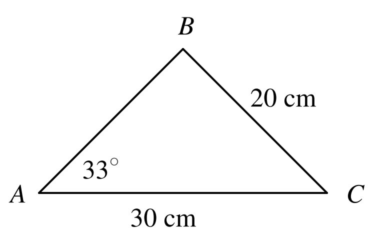


fig. 2.11

Using the sine rule,

$$\frac{20}{\sin 33}=\frac{30}{\sin B} \quad\Rightarrow\quad \sin B=\frac{20}{30}\times\sin 33=0.363$$

$\Rightarrow\quad B=\sin^{-1}0.363=21.3^\circ$.

So, the calculator tells us that $B=48.6^\circ$. However, this is not the whole story. Look back at the graph of $\sin(x)$, for example *fig. 2.5*. We can see that there is more than one angle that gives a sine of 0.75 (in fact, there are infinitely many such angles). Another such angle would be $180^\circ-48.6^\circ=131.4^\circ$ which we can deduce form the symmetry of the graph of $\sin(x)$. (Another angle which would give a sine of 0.75 is $360^\circ+48.6^\circ=408.6^\circ$, but since the angles in a triangle add up to 180°, this is not a valid consideration).

Let us continue under the assumption that $B=48.6^\circ$. Therefore,

$C=180-(33+48.6)=98.4^\circ$.

Then,

$$\frac{20}{\sin 30} = \frac{AB}{\sin 98.4} \quad \Rightarrow \quad AB = \frac{20}{\sin 30} \times \sin 98.4$$

$$\Rightarrow \quad AB = 39.6 \text{ cm}$$

But, if we had taken $B = 131.4^\circ$, then we would have $C = 18.6^\circ$ and $AB = 12.8$ cm (check).

Both sets of solutions are equally valid. This problem is **ambiguous**, there are two possible solutions. Be aware of this eventuality!

Test 2.4 Solve the triangle *DEF*, where $E = 81^\circ$, $F = 62^\circ$ and $d = 4$ m.

The Cosine Rule

Look at *fig. 2.8*. There is a relation involving the length of all of the sides and one of the angles. The relation is as follows:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

and is called the **cosine rule**.

A Proof of the Cosine Rule

Look back at *fig. 2.9*. We can see form triangle *ABX* that,

$$c^2 = h^2 + (AX)^2 \quad \textbf{(1)}$$

And from triangle *BCX* we can see that,

$$a^2 = h^2 + (CX)^2 \quad \textbf{(2)}$$

Rearranging **(1)** for h^2 and substituting into **(2)** gives:

$$a^2 = c^2 - (AX)^2 + (CX)^2 \quad \Rightarrow \quad c^2 = a^2 + (AX)^2 - (CX)^2.$$

Now, $(AX)^2 - (CX)^2$ is a difference of squares, and so can be factorised as $(AX + CX)(AX - CX)$. So, we have that:

$$c^2 = a^2 + (AX + CX)(AX - CX) \quad \textbf{(3)}$$

But:

$AX + CX = b$ ....**(4)**

and, $AX - CX = AX + CX - 2CX = b - 2CX$. But $CX = a\cos C$, so:

$AX - CX = b - 2a\cos C$ ....**(5)**

Substituting **(4)** and **(5)** into **(3)** gives:

$$c^2 = a^2 + b(b - 2a\cos C)$$

$$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos C \text{ as required.}$$

This can be rearranged to give an alternative form for finding an angle given all three sides:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 2.5 Solve the triangle *ABC*, where $AC = 7$ cm , $BC = 8$ cm and $C = 40^\circ$.

First make a sketch and mark on the given information, shown in *fig. 2.12*.

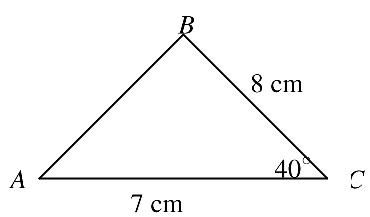


fig. 2.12

Using the cosine rule, we have:

$$(AB)^2 = 8^2 + 7^2 - 2\times 8\times 7\times \cos 40$$

$$\Rightarrow \quad AB = \sqrt{27.20} = 5.22 \text{ cm}.$$

Using the cosine rule a second time gives:

$$8^2 = 5.22^2 + 7^2 - 2\times 5.22\times 7\times \cos A$$

$$\Rightarrow \quad \cos A = \frac{5.22^2 + 7^2 - 8^2}{2\times 5.22\times 7} \quad \Rightarrow \quad A = 80.4^\circ .$$

Therefore, $B = 180^\circ - (40^\circ + 80.35^\circ) = 59.7^\circ$.

Test 2.5 Solve the triangle *XYZ*, where $XY = 9.5$ m , $YZ = 4$ m and $XZ = 7$ m .

Test 2.6 Solve the triangle *UVW*, where $VW = 88.3$ m, $UV = 97$ m and $U = 37^\circ$.

Look back at *fig. 2.9*. The area of this triangle is $\frac{1}{2}bh$. But, since $h = a\sin C$, we can write the area as:

$$\text{Area} = \frac{1}{2}ab\sin C$$

This formula allows us to find the area of any triangle given two sides and the included angle.

Test 2.7 Calculate the area of triangle *ABC*, where $BC = 6$ cm, $AC = 9$ cm and $C = 30^\circ$.

Some Trigonometric Identities

Look at *fig. 2.13*. We can see that $\sin\theta = \frac{b}{a}$, $\cos\theta = \frac{c}{a}$ and $\tan\theta = \frac{b}{c} = \frac{\sin\theta}{\cos\theta}$.

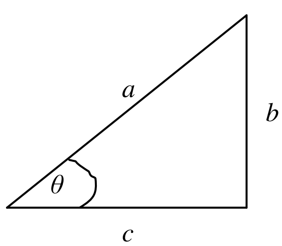


fig. 2.13

So we have the relationship:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

From Pythagoras' Theorem, we have that:

$b^2 + c^2 = a^2$.

Dividing throughout by a^2 gives:

$\left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 = 1$ or equivalently,

$$\sin^2\theta + \cos^2\theta = 1$$

This is an important trigonometric identity.
You may be asked to prove a trigonometric identity by using the above identities and some algebraic manipulation (see example 2.6). As a general rule, you should start with the most complicated side of the expression and, by using algebraic manipulation and the standard identities above, make it look like the other side of the expression.

Example 2.6 Prove the identity, $\sin\theta\tan\theta \equiv \frac{1}{\cos\theta} - \cos\theta$.

Now, we should start with the most complicated side, and make it look like the other side. Here though, both sides of the identity look equally simple. One obvious thing to do would be to rewrite $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. So let us work on the LHS:

$$\text{LHS} = \sin\theta\tan\theta = \sin\theta \times \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta}.$$

Now, using the identity, $\sin^2\theta + \cos^2\theta = 1$, we can see that, $\sin^2\theta = 1 - \cos^2\theta$. So we can write:

$$\text{LHS} = \frac{1-\cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \cos\theta = \text{RHS}.$$

So we have shown that the LHS equals the RHS, hence we have proved the identity, $\sin\theta\tan\theta \equiv \frac{1}{\cos\theta} - \cos\theta$.

> **Test 2.8** Using the two previously established trigonometric identities and algebraic manipulation, prove the identity, $\tan A + \frac{1}{\tan A} \equiv \frac{1}{\cos A} \times \frac{1}{\sin A}$.

Solving Triginometric Equations

Trigonometric identities are also useful for solving trigonometric equations.

Example 2.7 Solve the equation $2\sin\theta - \cos\theta = 0$ for the range $0 \le \theta \le 360^\circ$.

Rewrite the equation as, $2\sin\theta = \cos\theta$. Divide both sides by $\sin\theta$ to get, $2 = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$. So we have,

$\tan\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 26.6^\circ$ This is the answer that the calculator gives but beware, we were asked to look in the range $0 \le \theta \le 360^\circ$. Look back at the graph of $\tan\theta$ in *fig. 2.6*, we can see that there is another angle that has a tangent of $\frac{1}{2}$ in the given range, and it is $\theta = 180^\circ + 26.6^\circ = 206.6^\circ$.

So the **two** answers are $\theta = 26.6^\circ,\ 206.6^\circ$.

Example 2.8 Solve $2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \le \theta \le \pi$.

This is a quadratic equation in $\cos\theta$. To make this look like a more familiar form, we can make the substitution, $x = \cos\theta$. Now the equation becomes:

$$2x^2 - 3x + 1 = 0$$

which can be factorised as $(2x-1)(x-1) = 0 \quad \Rightarrow \quad x = \frac{1}{2}$ or $x = 1$. Now remember that $x = \cos\theta$, so we have:

$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 1.05 \quad$ or $\quad \cos\theta = 1 \quad \Rightarrow \quad \theta = 0$. These are the only solutions for the given domain.

Test 2.9 Solve $5\sin\theta = 2\cos\theta$ for $0 \le \theta \le 2\pi$. Note: this question is set in **radians**.

Test 2.10 Solve $2\sin^2\theta + 3\sin\theta = 2$ for values of θ between 0° and 360°.

2.2 Exponents and Logarithms

Laws of Indices

Recall the following rules of indices.

- $a^m a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

- $(a^m)^n = a^{mn}$
- $a^{-m} = \frac{1}{a^m}$
- $a^{\frac{1}{2}} = \sqrt{a}$
- $a^0 = 1$
-

The Graph of $y = a^x$

An **exponential function** is a function where a constant base is raised to a variable exponent, for example $f(x) = 2^x$.

Make a table of values for $f(x) = 2^x$ for $-2 \le x \le 5$ and use this to plot the function $f(x) = 2^x$ for the given domain. The plot should look like *fig. 2.14*.

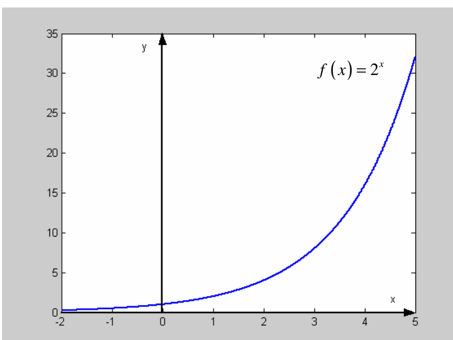


fig. 2.14

Notice that this graph is strictly increasing. This shape of graph is sometimes called **exponential growth**. More generally, the graph of $f(x) = a^x$ shows exponential growth for $a > 1$, and is sometimes used as a simple population model. Exponential growth also occurs as the limit of discrete processes such a compound interest.

Notice that the graph of $f(x) = a^x$ cuts the *y*-axis at $y = 1$.

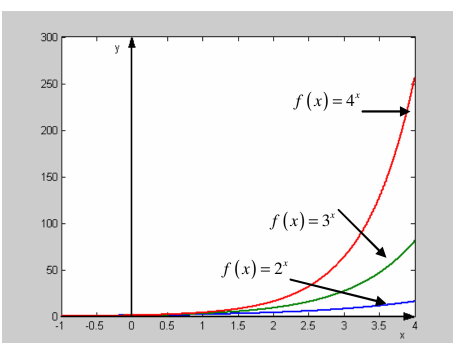


fig. 2.15

The graph of $f(x) = a^x$ has the same basic shape as *fig. 2.14* for all $a > 1$, but the rate of growth increases with increasing *a*, as illustrated in *fig. 2.15*.

Logarithms

If *a* is a positive real number other than 1, then the **logarithm of *x* with base *a*** is defined by:

$y = \log_a x \quad \Leftrightarrow \quad x = a^y$ (for every $x > 0$ and every real number y).

Thus, logarithms are simply an alternative way of writing exponents.

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y$$

For example the expression $100 = 10^x$ can be written in logarithmic form as $x = \log_{10} 100$. This is a logarithm base 10. Logarithms base 10 are often used and are usually written simply as 'log' rather than '$\log_{10}$'. Whenever you see 'log', assume that it is base 10. So, we would usually write $100 = 10^x \quad \Leftrightarrow \quad x = \log 100$.

Test 2.11 Write the following equations in logarithmic form.

a) $1000 = 10^x$ b) $108 = 2^y$ **c)** $729 = 3^x$**.**

On your calculator, you will notice the button 'LOG'. We have established that $100 = 10^x \quad \Leftrightarrow \quad x = \log 100$. If you press 100 followed by 'LOG', your calculator will return the answer '2', so we have found that $x = 2$. Looking back at the original expression, $100 = 10^x$, we can now see that $100 = 10^2$. This is a trivial example. Let us look at a less trivial case.

Example 2.9 Solve for *x*, $10^x = 150$.

We can not solve this equation by sight, but we can easily solve it using the idea of logarithms.

$10^x = 150 \quad \Leftrightarrow \quad x = \log 150 = 2.18$.

The Laws of Logarithms

If *x* and *y* are any two positive real numbers, then

- $\log_a (xy) = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$
- $\log_a (x^n) = n \log_a x$ for every real number *n*.

Let us prove the first Law of Logarithms. The others have similar proofs.

Let $p = \log_a x$ and $q = \log_a y$.

Then $a^p = x$ and $a^q = y$, from the definition of the logarithm.

Now, $xy = a^p a^q = a^{p+q}$, by the laws of indices.

So, $xy = a^{p+q}$ which can be written in logarithmic form as,

$\log_a xy = p + q$.

But since $p = \log_a x$ and $q = \log_a y$, the equation above can be written as:

$\log_a(xy) = \log_a x + \log_a y$ as required.

Solving Equations of the Form $a^x = b$

We solved an equation of this form in example 2.9, however, this was an artificial example and could not have solved in this way if the '10' had been something else, like a '4'.

Supposed we are asked to solve the equation $3^x = 4$ for x. The method we would use is to 'take logarithms' of both sides of the equation. Taking logarithms of both sides of the equation does not change the equation, in the same way that adding 2 to both sides of an equation does not change the equation. We can take logarithms of both sides to any base, but since we have a 'LOG' (base 10) button on our calculator, it seems practical to choose to take logs to base 10.

So, given $3^x = 4$, we take logarithms base 10 of both sides of the equation:

$\log(3^x) = \log 4$.

From the laws of logarithms, this is equivalent to:

$$x \log 3 = \log 4 \quad \Rightarrow \quad x = \frac{\log 4}{\log 3} = 1.26.$$

So we have found that $x = 1.26$.

Example 2.10 Solve the equation $6^{2x+3} = 29$ for x.

Taking logarithms base 10 of both sides of the equation gives:

$$\log\left(6^{2x+3}\right) = \log 29 \quad \Rightarrow \quad (2x+3)\log 6 = \log 29$$

$$\Rightarrow \quad 2x+3 = \frac{\log 29}{\log 6} \quad \Rightarrow \quad x = \frac{1}{2}\left(\frac{\log 29}{\log 6} - 3\right)$$

$$\Rightarrow \quad x = -0.56$$

Test 2.12 Solve the following equations for x:

a) $3^{2x} = \frac{1}{2}$ **b)** $12^{2x} = 3^{x}$.

2.3 Sequences and Series

Sequences

Definitions

A **sequence** is a list of numbers which follows a mathematical pattern. For example:

1, 3, 5, 7, 9, 11, …

is a sequence. Each number in the list is called an **element** of the sequence.

A sequence whereby there is a constant difference between consecutive terms is called an **arithmetic progression** (AP). The difference between consecutive terms of an AP is called the **common difference** and is often denoted by d. The example above is an arithmetic progression. The common difference of the above sequence is 2.

A sequence whereby each term is found by multiplying the previous term by a given factor is called a **geometric progression** (GP). For example:

1, 2, 4, 8, 16, 32, …

is a geometric progression. The factor by which each term is multiplied to generate the next term is called the **common ratio** and is often denoted by r. The common ration of the above sequence is 2.

The n^{th} term of a sequence

Sometimes, a sequence, $a_1,\ a_2,\ a_3,...$ may be defined by a formula for the n^{th} term.

Example 2.11 The n^{th} term of a sequence is given by $a_n = 2n - 3$. Write down the first 6 terms of the sequence. Is the sequence an arithmetic progression or a geometric progression? What is the common difference / common ratio of the series? Find the value of n for which $a_n = 163$.

We are asked for the first 6 terms. We find these by simply inserting 1,2,3,4,5,6 for n into the formula for the n^{th} term. To find the 1st term, we insert $n = 1$ into the formula for the n^{th} term. Doing this gives us:

$a_1 = (2 \times 1) - 3 = -1$.

To find the 2nd term, we insert $n = 2$ into the formula. Doing this gives us:

$a_2 = (2 \times 2) - 3 = 1$.

To find the 3rd term, we insert $n = 3$ into the formula. Doing this gives us:

$a_3 = (2 \times 3) - 3 = 3$.

Continuing in this way, we find the first 6 terms to be:

$-1,\ 1,\ 3,\ 5,\ 7,\ 9$.

To determine whether this sequence is arithmetic or geometric, we subtract consecutive terms to see if there is a common difference. If not, try dividing consecutive terms of the sequence to see if there is a common ratio.

In this case, we can see that there is a common difference of 2 between each of the terms. Therefore the sequence is an arithmetic progression with common difference 2.

Next we are asked to find the value of n for which $a_n = 163$. This is a matter of simple substitution. We want:

$a_n = 2n - 3 = 163$, so:

$$n = \frac{163 + 3}{2} = 83.$$

So the 83rd term is 163.

Test 2.13 Consider the sequence $9, 3, 1, \frac{1}{3}$. Is this sequence an AP or a GP? Write down the common difference / ratio. What is the 6th term?

Series

A **series** is the sum of the terms of a sequence. Like a sequence, a series may be finite or infinite. A series is called **arithmetic** if its terms follow an arithmetic progression. A series is called **geometric** if its terms follow a geometric progression.

1+2+3+…+100=?

Let us take a brief detour to consider a particular problem, which will help us in the next section.

What is $1+2+3+...+100$? This seems like a difficult problem at first sight, but with a little ingenuity there is a simple, short solution. There is more than one way of tackling this problem, but let us take the following approach.

Let. $S = 1+2+3+...+100$. We can rewrite this as:

$$S = 1+(1+1)+(1+2)+(1+3)+...+(1+99) \quad \textbf{(1)}$$

We can reorder this sum as:

$$S = (1+99)+(1+98)+(1+97)+(1+96)+...+1 \quad \textbf{(2)}$$

$$1+(1+1)+(1+2)+(1+3)+...+(1+99)$$

$$(1+99)+(1+98)+(1+97)+(1+96)+...+1$$

Notice that if we add each of the terms in **(1)** and **(2)** as indicated by the arrows opposite, each add to 101

So, **(1)** + **(2)** gives:

$$2S = \left[1+(1+99)\right]+\left[(1+1)+(1+98)\right]+\left[(1+2)+(1+97)\right]+...+\left[(1+99)+1\right]$$

$$\Rightarrow \quad 2S = 101+101+101+...+101 \qquad \text{(100 lots of 101)}$$

$$\Rightarrow \quad S = \frac{100\times 101}{2} = 5050.$$

So, we have found that $1+2+3+...+100=5050$.

It is reported that Carl Friedrich Gauss (1777 – 1855) solved this problem when presented with it at school at the age of seven. He went on to become one of the greatest mathematicians of all time.

Sum of an Arithmetic Sequence

Consider a general arithmetic sequence with first term *a*, common difference *d* and consisting of *n* terms. This can be represented by:

$$a, a+d, \ a+2d, \ a+3d, \ ... \ , a+(n-1)d .$$

Notice that the n^{th} term is $a+(n-1)d$.

The sum of these terms is given by:

$$S_n = a+(a+d)+(a+2d)+(a+3d)+ \ ... \ +(a+(n-1)d) \text{ ....}\mathbf{(1)}$$

This can be rewritten as:

$$S_n = (a+(n-1)d)+(a+(n-2)d)+(a+(n-3)d)+...+a \text{ ....}\mathbf{(2)}$$

Adding **(1)** and **(2)** gives:

$$2S_n = (2a+(n-1)d)+(2a+(n-1)d)+(2a+(n-1)d)+...+(2a+(n-1)d)$$

$$= n(2a+(n-1)d)$$

$$\Rightarrow \quad S_n = \frac{n}{2}(2a+(n-1)d).$$

For an arithmetic sequence with first term *a* and common difference *d*, the sum of the first *n* terms is given by:

$$S_n = \frac{n}{2}(2a+(n-1)d).$$

Example 2.12 Calculate the sum of the first 250 natural numbers.

So, we are looking for the value of $1+2+3+...+250$. This is an arithmetic series with first term 1 and common difference 1. The number of terms in the series is 250. We use the formula above with,

$a=1,\ d=1,\ n=250$.

Substituting the values in gives:

$$1+2+3+...+250=\frac{250}{2}\left(2\times 1+(250-1)\times 1\right)$$

$$\Rightarrow\quad 1+2+3+...+250=31375.$$

Example 2.13 The sum of the series $2+5+8+11+....+m$ is 6370. How many terms does this series have? What is the value of *m*?

This is an arithmetic series. We are told that $2+5+8+11+....+m=6370$. We can see that the first term is 2 and the common difference is 3. We do not know how many terms there are in the sequence, let us say that there are *n* terms in the sequence.

So we have that, $a=2,\ d=3$ and *n* is unknown. We also know that the sum of the first *n* terms, S_n, is 6370. Substituting these values into the formula gives:

$$6370=\frac{n}{2}\left(2\times 2+(n-1)\times 3\right)\quad\Rightarrow\quad 6370=2n+\frac{3n}{2}(n-1).$$

Multiplying throughout by 2 and expanding the bracket gives:

$12740=4n+3n^2-3n \quad\Rightarrow\quad 3n^2+n-12740=0$. This quadratic equation in *n* factorises to:

$$(3n+196)(n-65)=0\quad\Rightarrow\quad n=-\frac{196}{3}\ \text{ or }\ n=65.$$

Since *n*, the number of terms in the series, cannot be negative, we must have that $n=65$, i.e. there are 65 terms in the series.

The first term is 2. Each subsequent term has 3 added on to the previous term. *m* is the 65th term, therefore:

$$m=2+(3\times 65)=197.$$

Test 2.14 Calculate the sum of the first 100 odd numbers.

Test 2.15 Calculate the sum of the series $100+85+70+55+...-80$.

Sum of a Geometric Sequence

Consider a general geometric sequence with first term *a*, common ratio *r* and consisting of *n* terms. This can be represented by:

$a,\ ar,\ ar^2,\ ...,\ ar^{n-1}$.

Notice that the n^{th} term is ar^{n-1}.

The sum of these terms is given by:

$$S_n = a + ar + ar^2 + \ ...\ + ar^{n-1} \quad \textbf{(1)}$$

Multiply **(1)** throughout by *r*:

$$rS_n = ra + ar^2 + ar^3 + \ ...\ + ar^n \quad \textbf{(2)}$$

Subtracting **(1)** from **(2)** gives:

$$rS_n - S_n = ar^n - a$$

$$\Rightarrow \quad S_n(r-1) = a(r^n - 1)$$

$$\Rightarrow \quad S_n = \frac{a(r^n-1)}{r-1}$$

For a geometric sequence with first term *a* and common ratio *r*, the sum of the first *n* terms is given by:

$$\Rightarrow \quad S_n = \frac{a(r^n-1)}{r-1}.$$

Example 2.14 Use the formula above to calculate $2+4+8+16+32+...+256$.

This is a geometric series with first term 2 and common ratio 2 consisting of 8 terms. So we have, $a = 2,\ r = 2,$ and $n = 8$. Substituting these values into the formula we have:

$$S_8 = \frac{2\left(2^8 - 1\right)}{2-1} = 510.$$

Test 2.16 Consider the geometric series, $4 + 6 + 9 + 13\frac{1}{3} + \dots$. What is the common ratio for this series? Using the formula, find the sum of the first 25 terms correct to 6 significant figures.

Test 2.17 Consider the geometric series, $2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$. What is the common ratio for this series? What is the sum of the first 15 terms to five significant figures? (Does this series ever reach 4?)

Sum to Infinity of a Geometric Sequence

Recall that the sum of the first *n* terms of a geometric sequence is given by:

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}.$$

What happens to this quantity as *n* gets very large?

If *r* is greater than 1, then as *n* gets very large, r^n gets very, very large. If, on the other hand, *r* is greater than 0 but less than 1 (a fraction), then as *n* gets very large, r^n gets very, very small. So, for *r* greater than 0 but less than 1, the term r^n becomes negligible for large *n*. In fact, this is also true if *r* is less than 0 but greater than -1. In other words, if the *size* of *r* is greater than 0 but less than 1 (ignoring the sign), then r^n approaches 0 as *n* approaches infinity. We use the symbol, $|r| < 1$, to denote that the size of *r* is less than 1, ignoring the sign. $|r| < 1$ is equivalent to $-1 < r < 1$.

If a series gets arbitrarily close to a given value, *l*, as the number of terms increases to infinity, we say that the series **converges** to *l*.

A geometric series, $a + ar + ar^2 + \dots + ar^{n-1}$, converges when $|r| < 1$.

$$S_n \to \frac{a}{1-r} \quad \text{as} \quad n \to \infty \quad \text{provided } |r| < 1.$$

The limit $\frac{a}{1-r}$ is known as the **sum to infinity**, and is denoted S_∞.

Example 2.15 Find the sum to infinity of the series, $100 + 50 + 25 + ...$

We have $a = 100$ and $r = \frac{1}{2}$. Substituting into the formula for the sum to infinity we have:

$$S_\infty = \frac{100}{1 - \frac{1}{2}} = 200.$$

Test 2.18 Find the sum to infinity of the series $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + ...$

Sigma Notation

Rather than writing lengthy sums, for example $1 + 2 + 3 + ... + 100$, we have a compact notation that is often used, the so called *sigma notation*. $\sum$ is the Greek capital letter sigma. It used in mathematics to stand for a 'sum'.

For example,

Last term

$$1 + 2 + 3 + ... + 100 = \sum_{n=1}^{100} n$$

General term

First term

So, for example, we can write:

$$2 + 4 + 8 + 16 + ... + 2^{10} = \sum_{n=1}^{10} 2^n.$$

$$1+4+9+16+25+\ldots=\sum_{n=1}^{\infty} n^2$$

Test 2.19 Write down the following series in sigma notation.

a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{12}}$ **b)** $3+5+7+9+11+\ldots$ **c)** $1+8+27+64+\ldots+m^3$

2.4 Algebra and Functions

Laws of Indices

- $a^m \times a^n = a^{m+n}$ $\quad a^{-m}=\frac{1}{m}$ $\quad a^{\frac{1}{2}}=\sqrt{a}$
- $\frac{a^m}{a^n}=a^{m-n}$ $\quad a^{\frac{m}{n}}=\sqrt[n]{a^m}$ $\quad a^0=1$

Transformations of the Graph *y = f(x)*

Recall from Chapter 1 the section on 'Graph Transformations'. We saw that:

- The transformation $f(x) \;\to\; f(x)\pm a$ is a translation by a units parallel to the y-axis in the positive direction.
- The transformation $f(x) \;\to\; f(x\mp a)$ is a translation by a units parallel to the x-axis in the positive/negative direction.
- The transformation $f(x) \;\to\; af(x)$ is a stretch, parallel to the y-axis, by a factor a.
- The transformation $f(x) \;\to\; f(ax)$ is a stretch, parallel to the x-axis, by a factor $\frac{1}{a}$.

Example 2.16 Make a sketch of the function $f(x) = \sin x$ for the domain $0 \le x \le 2\pi$. On the same graph, sketch $f(2x) = \sin 2x$.

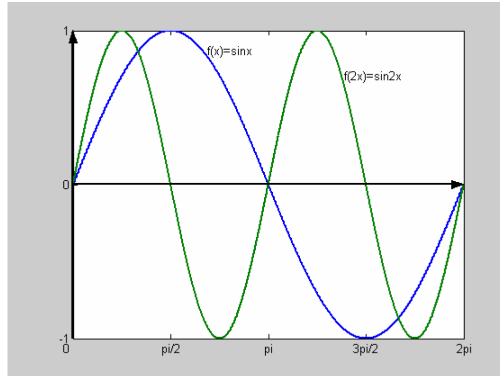


fig. 2.16

The sketches are shown in *fig. 2.16*. Notice that they both have the same basic shape and they both start and end at the same points. Notice also that the point $(\pi, 0)$ on the original graph, sin *x* is unaffected by thr transformation. The graph of $f(2x) = \sin 2x$ is similar to the graph of $f(x) = \sin x$, but it has been 'compressed' by a factor of 2, or stretched by a factor of ½ parallel to the *x*-axis.

This has the effect of halving the period. $f(x) = \sin x$ has a period of 2π, while $f(2x) = \sin 2x$ has a period of π.

Example 2.17 Make a sketch of the function $f(x) = \cos x$ for the domain $0 \le x \le 2\pi$. On the same graph, sketch $f\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$.

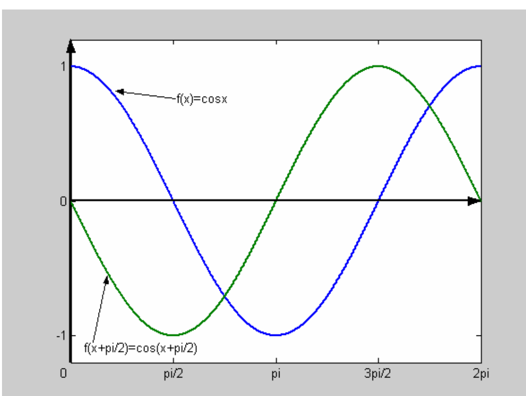


fig. 2.16

The graphs are shown in *fig. 2.16*. Remember that the transformation $f(x) \rightarrow f\left(x - \frac{\pi}{2}\right)$ has the effect of moving the graph of $f(x)$ parallel to the *x*-axis $\frac{\pi}{2}$ units in the positive direction.

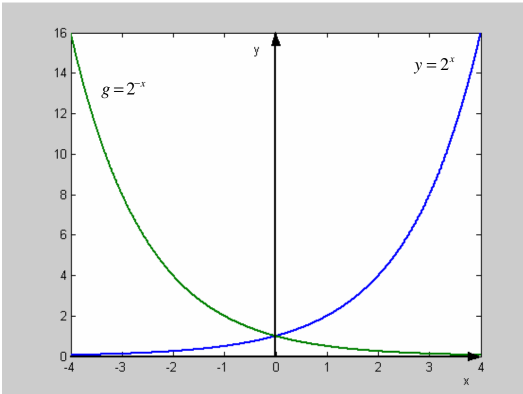


fig. 2.17

Example 2.18 Make a sketch of the function $y=2^x$ for the domain $-4 \le x \le 4$. On the same graph, sketch $g=2^{-x}$.

We have seen the graph $y=2^x$ before. We have not come across the transformation $f(x) \rightarrow f(-x)$ before. This is **not** a stretch or a translation. It is in fact a **reflection** in the *y*-axis. Also note that the transformation $f(x) \rightarrow -f(x)$ is a reflection in the *x*-axis.

Test 2.19 Make a sketch of the function $y=x^2$. On the same set of axes, sketch the functions $p=2x^2$ and $q=\frac{1}{2}x^2$ stating clearly how the curves relate to each other.

Test 2.20 Make a sketch of the function $y=3^x$. On the same set of axes, sketch the function $p=-3^x$ stating clearly how the curves relate to each other.

Test 2.21 Make a sketch of the function $f(x)=\sin x$ for the domain $0 \le x \le 2\pi$. On the same set of axes, sketch the function $f\left(x+\frac{\pi}{2}\right)=\sin\left(x+\frac{\pi}{2}\right)$ stating clearly how the curves relate to each other.

2.5 Differentiation

Recall From Chapter 1 that, if $y=ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x}=nax^{n-1}$, which is valid for all $a,\ n \in \mathbb{R}$. Also recall the laws or indices from section 2.2. When differentiating a function, we always aim to write each term of the function in the form ax^n or $ax^{\frac{m}{n}}$, where the index and / or the coefficient may be positive or negative. For example, when asked to differentiate $\frac{1}{x}$, we first write this as x^{-1}.

Example 2.19 Given that $y=\frac{x+\sqrt{x}}{\sqrt{x^3}}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

First we need to rearrange *y* into a more convenient form to work with.

$$y=\frac{x+\sqrt{x}}{\sqrt{x^3}}=\frac{x}{x^{\frac{3}{2}}}+\frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}}}=x^{1-\frac{3}{2}}+x^{\frac{1}{2}-\frac{3}{2}}$$

$$=x^{-\frac{1}{2}}+x^{-1}.$$

Now we can easily differentiate this term by term:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}x^{-\frac{3}{2}}-x^{-2}=-\frac{1}{2\sqrt{x^3}}-\frac{1}{x^2}.$$

So, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2\sqrt{x^3}}-\frac{1}{x^2}$.

Example 2.20 Differentiate $f\left(x\right)=x\sqrt{x^2}+\frac{2}{x^2}-\frac{\sqrt[3]{x}}{\sqrt{x}}$.

First we need to rearrange *y* into a more convenient form to work with.

$$f\left(x\right)=x\sqrt{x^2}+\frac{2}{x^2}-\frac{\sqrt[3]{x}}{\sqrt{x}}=x.x^{\frac{1}{2}}+2x^{-2}-x^{\frac{3}{2}}.x^{-\frac{1}{2}}$$

$=x^{\frac{3}{2}}+2x^{-2}-x$. Now we can easily differentiate this term by term:

$$f'\left(x\right)=\frac{3}{2}x^{\frac{1}{2}}-4x^{-1}-1=\frac{3}{2}\sqrt{x}-\frac{4}{x}-1.$$

So, $f'\left(x\right)=\frac{3}{2}\sqrt{x}-\frac{4}{x}-1$.

Test 2.22 If $y=\left(x+2\right)^2$, calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Test 2.23 If $f(x)=\dfrac{x+1}{\sqrt{x}}$, calculate $f'(x)$ and $f''(x)$.

Test 2.24 Sketch a graph of the function $f(x)=\dfrac{1}{x}$. Calculate the gradient of the tangent to this function at the point $x=-2$.

2.6 Integration

Recall from Chapter 1 that $\int ax^n \, \mathrm{d}x = \dfrac{a}{n+1}x^{n+1}+c$, which is valid for all $a,\ n \in \mathbb{R}$. As in the previous section on differentiation, we also need to have the laws of indices at the front of our minds when working on integration problems. Also as in the previous section, when integrating functions we always aim to write each term of the function in the form ax^n or $ax^{\frac{m}{n}}$, where the index and / or the coefficient may be positive or negative.

Example 2.21 Calculate the indefinite integral, $\displaystyle\int \frac{x+\sqrt[3]{x}}{\sqrt{x^3}}\,\mathrm{d}x$.

We can see that,

$\dfrac{x+\sqrt[3]{x}}{\sqrt{x^3}} = x^{-\frac{1}{2}}+x^{-\frac{7}{6}}$. (check) So,

$$\int \frac{x+\sqrt[3]{x}}{\sqrt{x^3}}\,\mathrm{d}x = \int x^{-\frac{1}{2}}+x^{-\frac{7}{6}}\,\mathrm{d}x$$

$$= 2x^{\frac{1}{2}}-6x^{-\frac{1}{6}}+c = 2\sqrt{x}-\frac{6}{\sqrt[6]{x}}+c.$$

So, $\displaystyle\int \frac{x+\sqrt[3]{x}}{\sqrt{x^3}}\,\mathrm{d}x = 2\sqrt{x}-\frac{6}{\sqrt[6]{x}}+c$.

Test 2.25 Calculate the indefinite integral, $\int (x+2)^2 \, \mathrm{d}x$. Hence calculate the definite integral $\int^{1} (x+2)^2 \, \mathrm{d}x$

GCE AS/A Mathematics Chapter 3

3.1 Algebra and Functions

Definition of a Function

So far we have discussed functions in general terms without specifying exactly what a function is. Before we give a formal definition of a function, let us introduce some new language and concepts concerning functions.

Functions are sometimes called 'mappings', and we may think of a function as a mapping from one set to another. This is illustrated below for the function $f(x) = x + 2$.

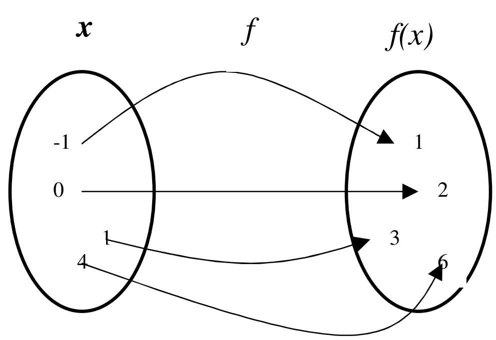


fig. 3.1

For example, f 'sends' -1 to 1 (or -1 is mapped to 1), f 'sends' 4 to 6 (or 4 is mapped to 6) etc.
We say that '1 is the image of -1 under f', '6 is the image of 4 under f' etc.

Domain and range

The set of all numbers that we can feed into a function is called the **domain** of the function. The set of all numbers that the function produces is called the **range** of a function. Often when dealing with simple algebraic function, such as $f(x) = x + 2$, we take the domain of the function to be the set of real numbers, $\mathbb{R}$. In other words, we can feed in any real number x into the function and it will give us a (real) number out. Sometimes we *restrict the domain*, for example we may wish to consider the function $f(x) = x + 2$ in the interval $-2 \le x \le 2$.

Consider the function $f(x) = x^2$. What is the range of $f(x)$? Are there any restrictions on the values that this function can produce? When trying to work out the range of a function it is often useful to consider the graph of the function, this is shown in *fig. 3.2*.

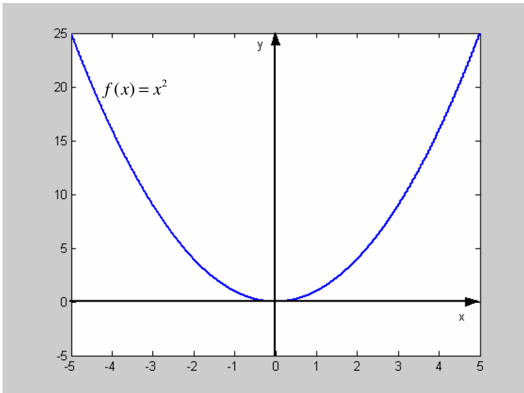


fig. 3.2

We can see that the function only gives out positive numbers (x^2 is always positive for any real number *x*). There are no further restrictions. We can see that f can take any positive value, therefore the range of f is the set of all positive numbers, we may write $f(x) \geq 0$.

One-to-one functions

When each of the elements of the domain is mapped to a *unique* element of the range, under a mapping, the mapping is said to be **one-to-one**. When two or more elements of the domain are mapped to the *same* element of the range under a mapping, the mapping is said to be *many-to-one*. Below are two examples. The function f is one-to-one, the function g is many-to-one.

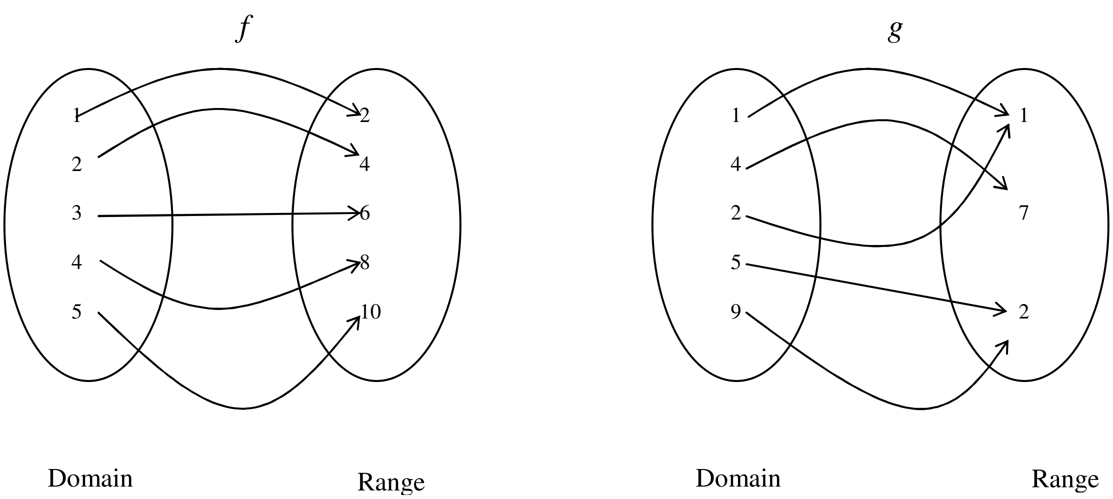


fig. 3.3

Are all algebraic expressions functions?

We need to define more precisely what we mean by a 'function'. We can define a function as a rule that uniquely associates each and every member of one set with a member or members of another set. This means that every element of the domain is mapped to an element of the range such that the image of any element in the domain is unique. In other words, **each and every element of the domain must be mapped to one and only one element of the range**.

For example, consider the expression $y = \pm\sqrt{x}$. This is plotted in *fig 3.4*.

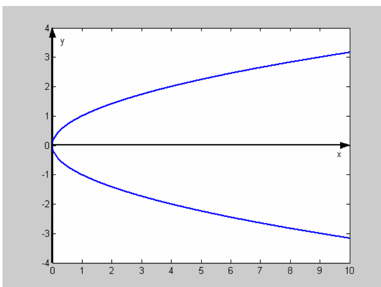


fig. 3.4

Notice that any value of x in the domain, except $x = 0$, (i.e. any positive real number) is mapped to *two* different values in the range. Therefore $y = \pm\sqrt{x}$ is **not** a function.

Test 3.1 Decide if the following (**1.** – **4.**) are functions. Justify your answers. In the cases that are functions, state the domain and the range.

1. $f(x)$ as shown in the graph in *fig*. 3.5.

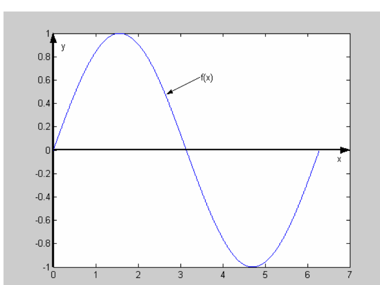


fig. 3.5

2. $x^2 + y^2 = 36$.

3. p as defined in *fig 3.6*.

4. φ as defined in *fig 3.7*.

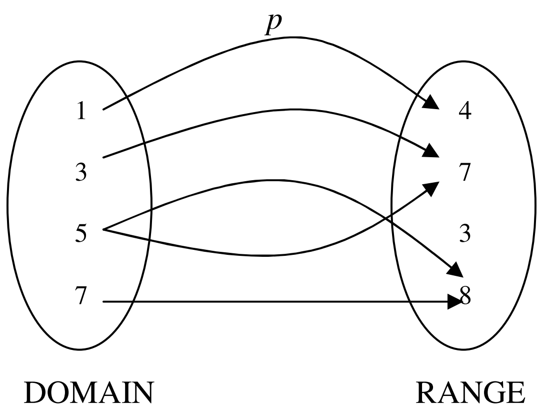


fig. 3.6

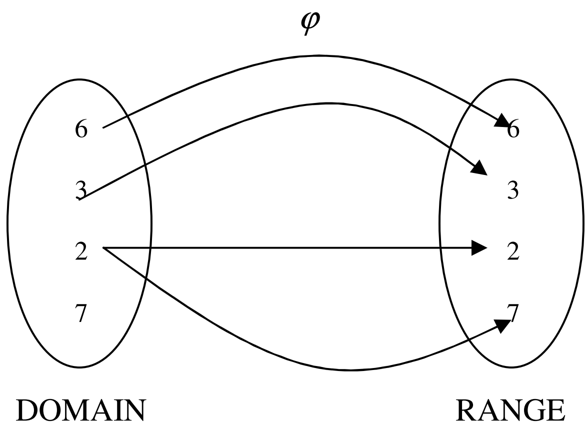


fig. 3.7

Composition of Functions

Consider the function, $g(x)=(x-2)^2$. If we were given a set of numbers and asked to perform the function g on each of them, we would have to carry out two separate calculations on any one of the given numbers; first we would have to subtract two from the number, then we would square the result. Thus, we may think of the function g as two functions in one. g is composed of the functions $p(x)=x-2$ and $q(x)=(p(x))^2$. We say that g is a **composite** function, and we write $g(x)=q(x)\circ p(x)$, or simply $g=p\circ q$.

! Notice that when calculating $g=p\circ q$, we **first perform *q*** and then perform p on the result.

! $g = p \circ q$ is sometimes written as $g = pq(x)$. This does **not** mean that we multiply the functions *p* and *q* together, it is simply an alternative notation for $g = p \circ q$.

Example 3.1 Work out $f \circ g$ given that $f(x) = \sqrt{x}$ and $g(x) = 2x + 7$.

$f \circ g$ means, first perform the function *g* and then perform the function *f* on the result. So given any number, the effect of $f \circ g$ is to multiply it by two, add 7 and then find the square root of the result, i.e.:

$f \circ g = \sqrt{2x+7}$.

Example 3.2 Given that $f = \sin 2x$ and $g = x^2$, find $f \circ g$ and $g \circ f$.

Given any real number, the composite function $f \circ g$ has the effect of first squaring the number and then finding the sine of two times the result, i.e.:

$f \circ g = \sin\left(2x^2\right)$.

Given any real number, the composite function $g \circ f$ has the effect of first finding the sine of two times the number and then squaring the result, i.e.:

$g \circ f = \left(\sin(2x)\right)^2$ or, in more conventional notation, $g \circ f = \sin^2(2x)$.

Example 3.3 If $p(x) = \frac{1}{x}$ and $q(x) = 4x - 8$, find suitable domains for the composite functions $p \circ q$ and $q \circ p$.

Now, $p \circ q = \frac{1}{4x-8}$. The important thing to remember here is that we cannot divide by zero, so we cannot have that $4x - 8 = 0$, i.e. we cannot have that $x = 2$. All other real numbers are valid as the domain of $p \circ q$, so we have that the domain of $p \circ q$ is $x \in \mathbb{R}$, $x \neq 2$.

$q \circ p = \frac{4}{x} - 8$. All real numbers except $x = 0$ are valid for this composite function, i.e. the domain is $x \in \mathbb{R}$, $x \neq 0$.

Test 3.2 If $f(x) = x + 5$ and $g(x) = +\sqrt{x}$, find $f \circ g$ and $g \circ f$.

Test 3.3 If $f(x)=x^2-4$ and $g(x)=\frac{1}{x}$, find suitable domains for the composite functions $f\circ g$ and $g\circ f$.

Test 3.4 If $f(x)=x-2$ and $g(x)=x^2$ and $h(x)=\frac{1}{x}$, find $h\circ f\circ g$.

Inverse Functions and their Graphs

Consider the simple, linear function $f(x)=3x-27$. If we feed $x=2$ into this function, we get out $f(2)=-21$. Suppose that we are told that the function has produced the number 9, but we do not know what input produced this number. We can easily work out the input number:

$$f(n)=3n-27=9 \quad \Rightarrow \quad n=\frac{9+27}{3}=12.$$

If we know the output of a given function and we require the input of the function, we can find it by using the **inverse function**.

We have that,

$$f(x)=3x-27 \quad \Rightarrow \quad x=\frac{f(x)+27}{3} \text{ ....(†)}$$

Now, given any output, $f(x)$, we can always find the input, *x*, using the above formula. The above formula reverses the effect of the original function. This is called the inverse function. We denote the original function by $f(x)$ and we denote the inverse function by $f^{-1}(x)$. We usually replace '*x*' with '$f^{-1}(x)$' and '$f(x)$' with '*x*' in (†) so that we have:

$$f^{-1}(x)=\frac{x+27}{3}.$$

We can think of a 'function machine' which takes an input, performs the function on it and produces an output. The inverse function machine takes the output from the original function and gives us the original input number, as illustrated in *fig. 3.8*.

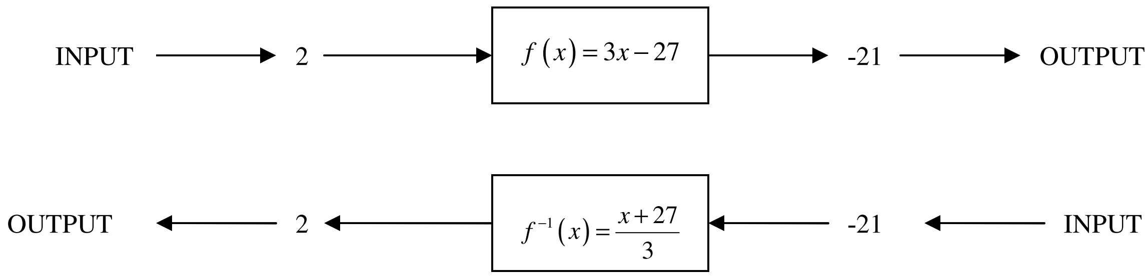


fig. 3.8

The notation $f^{-1}(x)$ simply stands for the inverse function of $f(x)$. It does **NOT** mean $f^{-1}(x) = \dfrac{1}{f(x)}$

So, we can see that the set of all inputs for $f(x)$, which we call the domain of $f(x)$, becomes the set of all outputs for $f^{-1}(x)$, which we call the range of $f^{-1}(x)$.

Example 3.4 If $f(x) = x + 2$, find $f^{-1}(x)$. On the same set of axes, plot $f(x)$ and $f^{-1}(x)$. What is the relationship between the two graphs?

Let $y = x + 2 \quad \Rightarrow \quad x = y - 2 \quad \Rightarrow \quad f^{-1}(x) = x - 2$.

The plots of $f(x)$ and $f^{-1}(x)$ are shown in *fig. 3.9*.

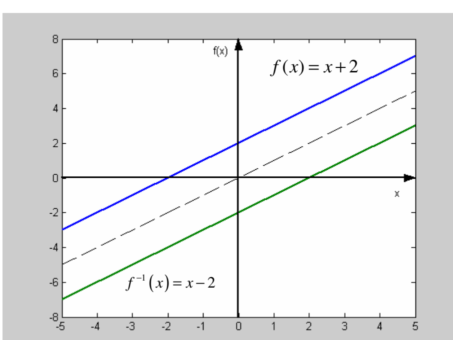


fig. 3.9

We can see from *fig. 3.9* that the graph of $f^{-1}(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$.

In fact, this is a general result for any invertible function (a function that has an inverse). Note that not all functions are invertible. **Only one-to-one functions are invertible.**

To illustrate why many-to-one functions are not invertible, consider the many-to-one function θ as illustrated in *fig. 3.10*.

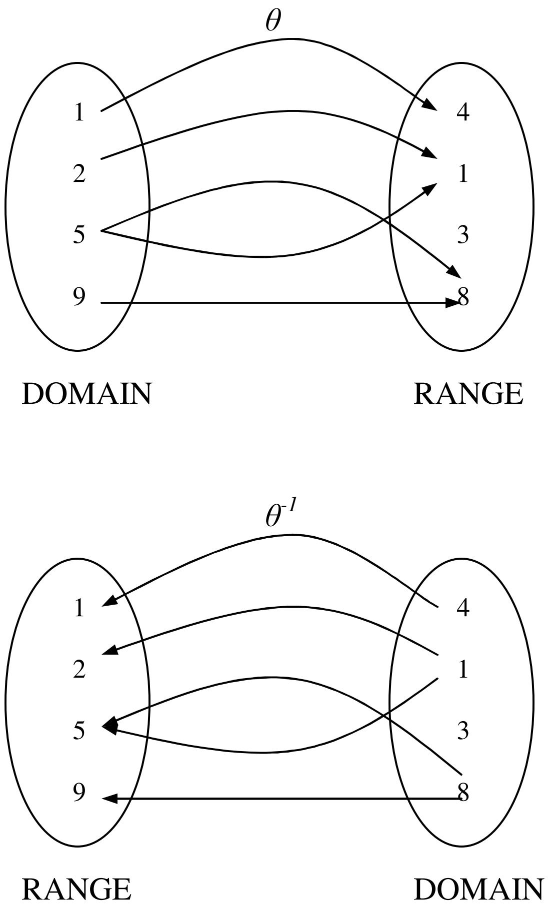


fig. 3.10

The inverse ‘function’ is illustrated in *fig. 3.10*. We can see that θ^{-1} is not a function, as it does not satisfy the definition. For example, 8 in the domain is sent to both 5 and 9 in the range, which is not allowed.

We conclude, therefore tat **only one-to-one functions are invertible.**

Example 3.4 If $f(x)=3x-5$, find the inverse function, $f^{-1}(x)$. On the same set of axes, plot $f(x)$ and $f^{-1}(x)$.

Let $y=3x-5$, then $x=\dfrac{y+5}{3}$ which means that $f^{-1}(x)=\dfrac{x+5}{3}$. The graphs of $f(x)$ and $f^{-1}(x)$ are shown in *fig. 3.11*.

Test 3.5 The function $f(x)=\dfrac{2x-6}{3+x}$ with domain, $x>-3$ is plotted in *fig. 3.12*. Calculate the formula for the inverse function, $f^{-1}(x)$. State the domain for which $f^{-1}(x)$ is defined. Use *fig. 3.12* to help you sketch the graph of $f^{-1}(x)$ stating clearly where the graph cuts the *x* and *y* axes.

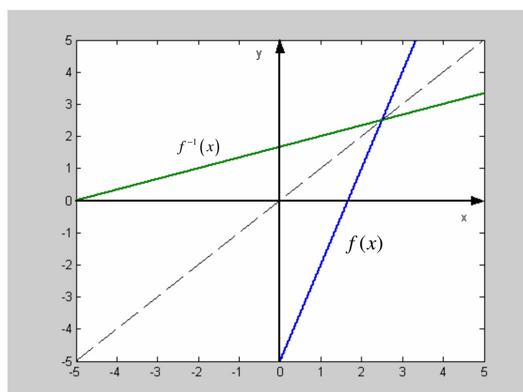


fig. 3.11

Test 3.6 Can you think of a function which is its own inverse? Can you think of two?

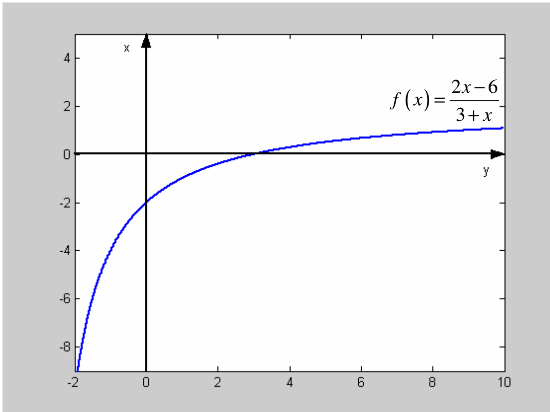


fig. 3.12

Suppose we have a function, $f(x)$, and we have calculated the inverse function, $f^{-1}(x)$. It is clear that if we perform the function *f* on any number, followed by the inverse function, f^{-1}, we will have the original number we started with. In mathematical language, $f^{-1}(x)\circ f(x)=x$, or, $f^{-1}(f(x))=x$. To see this in action, verify for example 3.4.

The Modulus Function

$|2-3|=1$, $|0-5|=5$, $|-2|=2$, $|1+7|=8$. The modulus sign, $|\ |$ indicates that we take the *absolute* value of the expression inside the modulus sign, i.e. all values are positive. We can define:

$$|x|=\begin{cases} x & \text{for } x\geq 0 \\ -x & \text{for } x<0 \end{cases}$$

Let us consider the graph of $y=|x|$. As we have said, $|x|$ is always positive, so the graph of $y=|x|$ cannot exist below the *x*-axis. For positive *x*, the graph of $y=|x|$ is the same as the graph of $y=x$; but for negative *x*, the graph of $y=|x|$ is the line $y=-x$. This is illustrated in *fig. 3.13*.

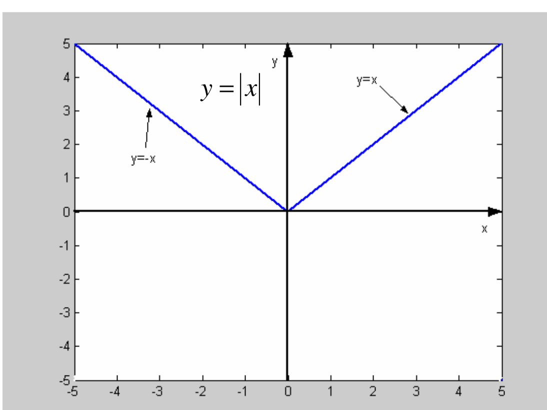


fig. 3.13

Note that the graph of $y=|x|$ is similar to the graph of $y=x$ except that the negative region of the graph is reflected in the *x*-axis.

In general, the graph of $y=|f(x)|$ is similar to the graph of $y=f(x)$ except that the negative region of the graph is reflected in the *x*-axis.

Example 3.5 Sketch the graph of $y = \left| -\frac{1}{2}x - 4 \right|$. State where this graph cuts the *y*-axis.

We begin by sketching the graph of $y = -\frac{1}{2}x - 4$, as shown in *fig. 3.14*.

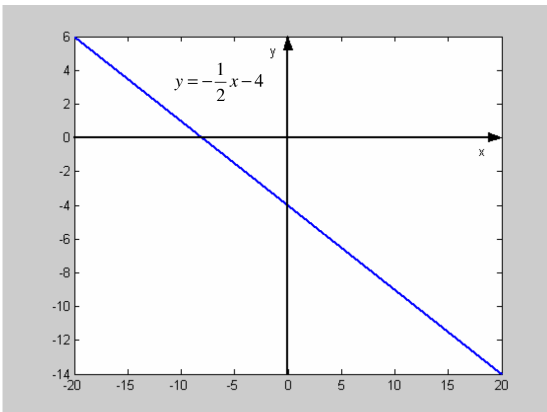


fig. 3.14

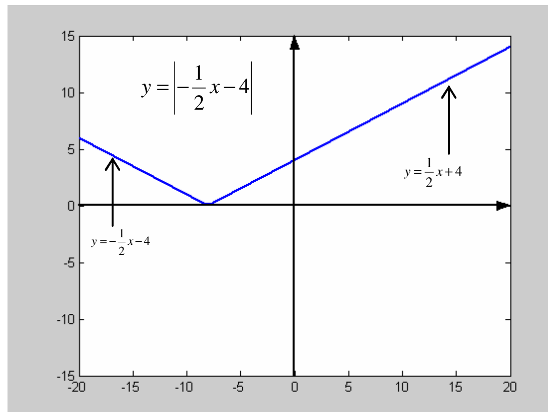


fig. 3.15

The portion of the graph below the *x*-axis is reflected in the *x*-axis to give us the graph of $y = \left| -\frac{1}{2}x - 4 \right|$. This reflected line has the equation $y = -\left(-\frac{1}{2}x - 4 \right) = \frac{1}{2}x + 4$. The complete graph of $y = \left| -\frac{1}{2}x - 4 \right|$ is shown in *fig. 3.15*.

The graph of $y = \left| -\frac{1}{2}x - 4 \right|$ cuts the *y*-axis when $y = \frac{1}{2}x + 4$ cuts the *y*-axis, i.e. at $y = 4$.

Example 3.6 Sketch the graph of $y = \left| x^2 - 5 \right|$. State where $y = \left| x^2 - 5 \right|$ cuts the *y*-axis.

We start by making a sketch of the graph $y = x^2 - 5$, shown in *fig. 3.16*. The negative region of the graph is reflected in the *x*-axis to give the complete graph of $y = \left| x^2 - 5 \right|$, as shown in *fig. 3.17*. The reflected portion of the graph has equation $y = -\left(x^2 - 5 \right) = -x^2 + 5$, and cuts the *y*-axis at $y = 5$.

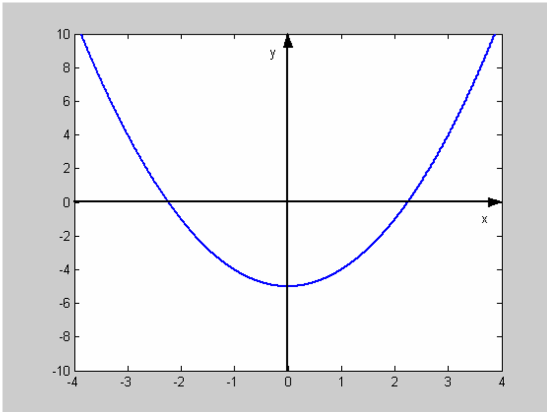


fig. 3.16

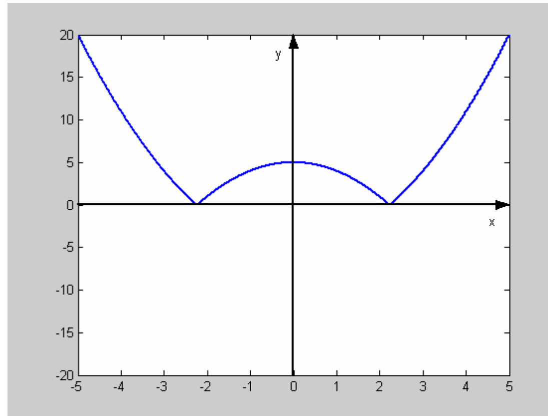


fig. 3.17

Test 3.7 Sketch $y = |\sin x|$ for $-\pi \le x \le \pi$.

Test 3.8 On the same set of axes, sketch the graphs of $y_1 = \left|\frac{1}{2}x + 2\right|$ and $y_2 = |2x - 7|$. Hence solve the equation $\left|\frac{1}{2}x + 2\right| = |2x - 7|$.

Test 3.9 Given that $f(x) = 3x - 146$, solve the equation $|f(x)| = 1$. *Hint:* a sketch may be useful.

Example 3.7 Solve the inequality $|3x - 3| < 4$.

Our first task is to sketch $y = |3x-3|$, this is shown in *fig. 3.18*.

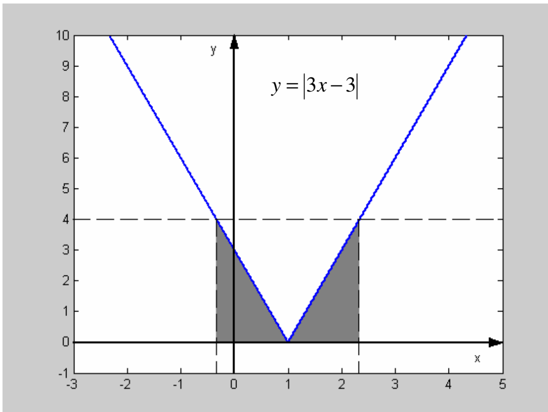


fig. 3.18

We then solve the equation $|3x-3| = 4$. To do this we must solve $3x-3=4$ and $-3x+3=4$. Doing this gives $x = 2\frac{1}{3}$ and $x = -\frac{1}{3}$. Graphically, these two points correspond to the points on the *x*-axis where the line $y = 4$ cuts the graph of $y = |3x-3|$.

We have then that $|3x-3| < 4$ when $-\frac{1}{3} < x < 2\frac{1}{3}$. Notice that the points $x = 2\frac{1}{3}$ and $x = -\frac{1}{3}$ are not included, as the original inequality is 'strictly less than'. The required region is shaded in *fig. 3.18*.

Test 3.10 Sketch the function $y = |x^2 - 22|$. Solve the inequality $|x^2 - 22| \le 3$, giving your answers in surd form where necessary. Shade the region $|x^2 - 22| \le 3$ on your sketch.

Combinations of Graphical Transformations

Recall the sections on 'graph transformations' from Chapters 1 and 2. In previous sections, we have considered the effect of one single transformation on a function. We can perform more than one transformation on a function as illustrated in example 3.9.

Example 3.9 By performing transformations of the graph of $y = x^2$, sketch the graph of $y = (x-2)^2 - 5$. We can easily sketch the graph of $y = x^2$, which is shown in *fig. 3.19*. We then perform the transformation $f(x) \rightarrow f(x-2)$ which, as we have seen from Chapter 1 is a translation parallel to the *x*-axis by 2 units in the positive direction. This gives us the graph of $y = (x-2)^2$, which is shown in *fig. 3.20*. Finally, we take the graph of $y = (x-2)^2$ and perform the transformation $f(x) \rightarrow f(x) - 5$, which is a translation parallel to the *y*-axis by 2 units in the negative direction. This gives us the graph of $y = (x-2)^2 - 5$, which is shown in *fig. 3.21*.

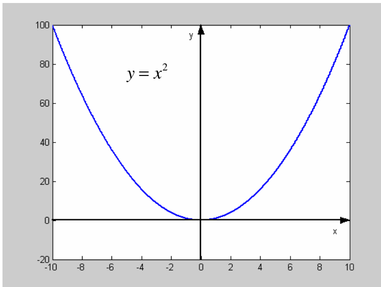


fig. 3.19

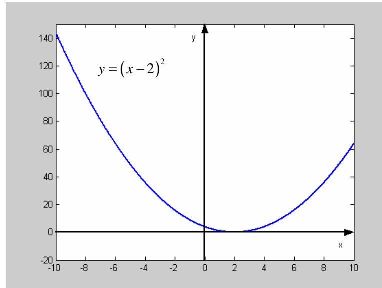


fig. 3.20

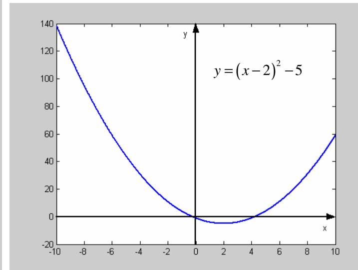


fig. 3.21

Test 3.11 By performing transformations of the graph of $y = x^2$, sketch the graph of $y = 3(x+2)^2$.

Example 3.9 By performing transformations of the graph of $y = x^2$, sketch the graph of $y = x^2 - 4x + 8$.

Remember that we can express any quadratic equation in the form $y = a(x \pm b)^2 \pm c$, where *a, b* and *c* are constants. We write $y = x^2 - 4x + 8$ as $y = (x-2)^2 + 4$. We can now sketch the required graph by performing the transformation $f(x) \rightarrow f(x-2)$, followed by the transformation $f(x) \rightarrow f(x)+4$ on the graph of $y = x^2$. The process is similar to example 3.8. We will skip intermediate steps and plot the end result, which is shown in *fig. 3.22*.

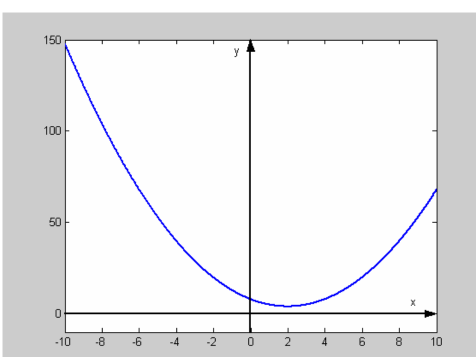


fig. 3.22

Test 3.12 By performing transformations of the graph of $y = x^2$, sketch the graph of $y = x^2 + 6x + 6$.

Test 3.13 Sketch the graph of $f(x) = x^2$. On the same axes, sketch the graph of $f(x-2)+1$.

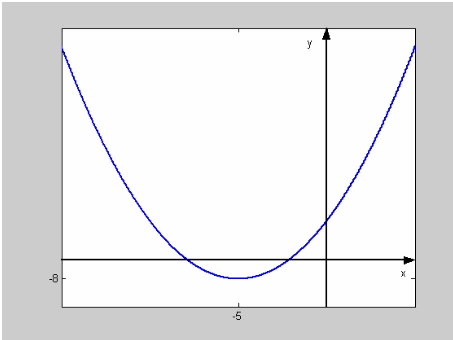


fig. 3.23

Test 3.14 The graph of $y = x^2$ has been subjected to two transformations to produce the graph shown in *fig. 3.23*. The minimum value of the graph in *fig. 3.23* occurs at $(-8, -5)$. State the two transformations that would produce *fig. 3.23* from $y = x^2$.

3.2 Trigonometry

Inverse Trigonometric Functions

We are already used to working with inverse trigonometric functions. Consider *fig. 3.24*, to find θ, we use the inverse sine function, $\theta = \sin^{-1}\left(\frac{1}{2}\right)$

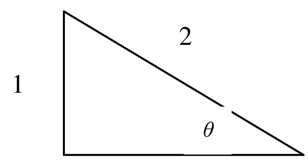


fig 3.24

$\Rightarrow \quad \theta = 30^\circ$ or $\frac{\pi}{6}$ radians. Recall the graph of sin*x* from Chapter 2, *fig. 2.3*, it is shown again below.

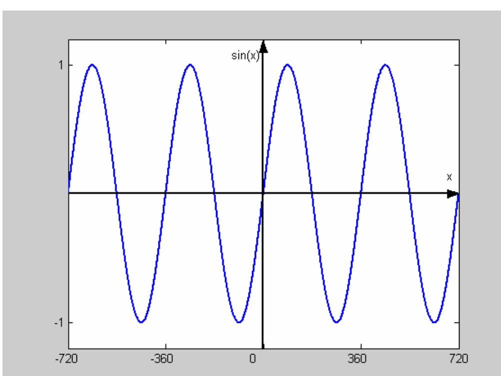


fig. 2.3

We can see that there are many (infinitely many) angles that have a sine of $\frac{1}{2}$, not just $\frac{\pi}{6}$ that our calculator tells us. Recall from the section 'inverse functions and their graphs, Chapter 3, that only one-to-one functions have inverses. The graph of $\sin x$ as shown opposite is not one-to one and so does not have an inverse. To proceed to define the inverse sine function, we must **restrict the domain** of $\sin x$ to $[-\pi/2,\ \pi/2]$.

The sine function with the restricted domain $x \in [-\pi/2,\ \pi/2]$ is shown in *fig. 3.25*.

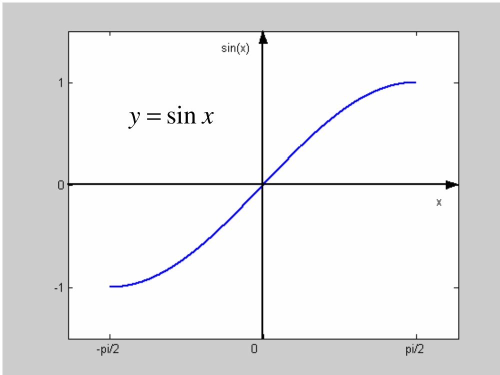


fig. 3.25

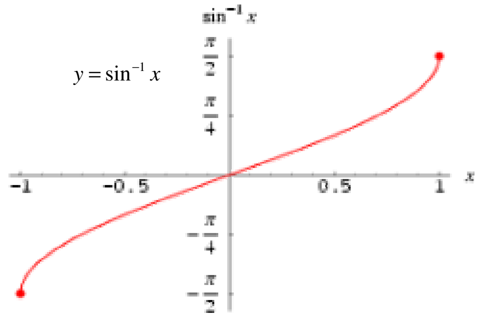


fig. 3.26

The sine function on this restricted domain is now one-to-one and so the inverse function, $y = \sin^{-1} x$ exists. Recall that the graph of the inverse function is obtained by reflecting the original function in the line $y = x$. The inverse sine function is shown in *fig. 3.26*. Notice that the domain of the sine function becomes the range of the inverse sine function, and that the range of the sine function becomes domain of the inverse sine function. Notice, the inverse sine function may be denoted by $\sin^{-1}$ or by arcsine.

! $\sin^{-1} x$ does **NOT** mean $\dfrac{1}{\sin x}$. $\sin^{-1}$ and arcsine are equivalent symbols for the inverse sine function.

We can restrict the domains of the cosine and tangent functions in a similar way so that their inverse functions can be defined. The cosine function is restricted to the domain $x \in [0,\ \pi]$ and the tangent function is restricted to the domain $x \in [-\pi/2,\ \pi/2]$. The graphs of cosine and tangent with restricted domains, along with their respective inverse functions are shown in the following graphs.

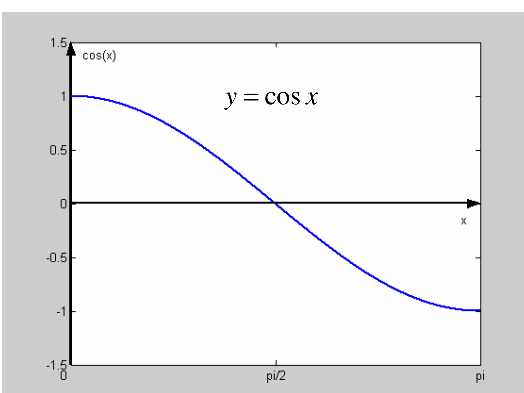


fig. 3.27

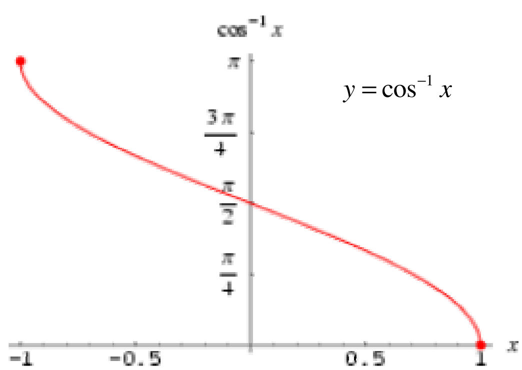


fig. 3.28

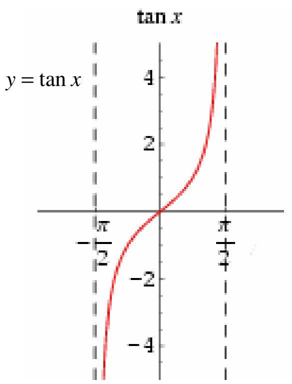


fig. 3.29

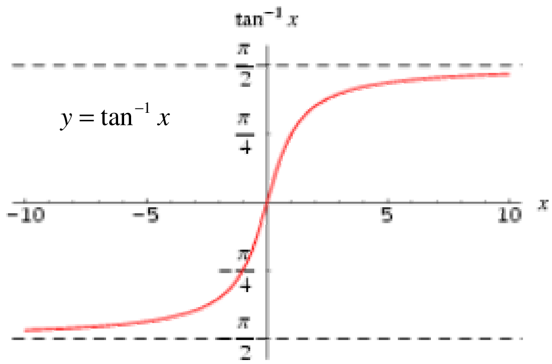


fig. 3.30

Note, alternative names for the inverse cosine function are $\cos^{-1}$ and arccos. Alternative names for the inverse tangent function are $\tan^{-1}$ and arctan.

Secant, Cosecant and Cotangent

We define the cosecant (cosec), secant (sec) and cotangent (cot) as follows:

$$\text{cosec}\,\theta \equiv \frac{1}{\sin\theta} \qquad \sec\theta \equiv \frac{1}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

Let us consider the graph of $\text{cosec}\,\theta$. By looking at the graph of $\sin\theta$ (*fig. 2.3*), we can see how the graph of $\text{cosec}\,\theta$ will look. For small positive values of θ, $\sin\theta$ is very small, so $\text{cosec}\,\theta \equiv \frac{1}{\sin\theta}$ will be very large. At $\theta = \frac{\pi}{2}$, $\sin\theta$ is 1, so $\text{cosec}\,\theta$ is also 1. When θ is close to π, $\sin\theta$ is close to zero, so $\text{cosec}\,\theta$ will be very large. We now have an idea of what the graph of $\text{cosec}\,\theta$ looks like in the range $0 \le \theta \le \pi$. Performing a similar analysis for other values of θ gives us the graph of $\text{cosec}\,\theta$ as shown in *fig. 3.31*.

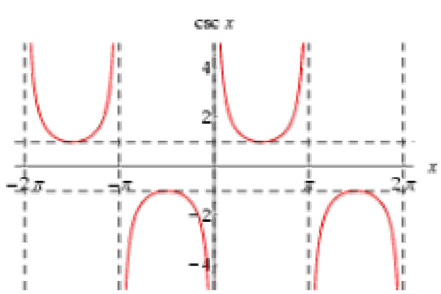


fig. 3.31

- Notice that $\operatorname{cosec}\theta$ is defined for all values of θ except $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$
- The function $f(\theta) = \operatorname{cosec}\theta$ is periodic, of period 2π.
- Also notice that for $f(\theta) = \operatorname{cosec}\theta$, $f(\theta) \geq 1$ or $f(\theta) \leq 1$. In other words, $|f(\theta)| \geq 1$.

By similar analysis, we can plot graphs for $\sec\theta$ (*fig. 3.32*) and $\cot\theta$ (*fig. 3.33*). <u>Task</u>: Write down analogous statements to the three bullet points above for the functions $f(\theta) = \sec\theta$ and $f(\theta) = \cot\theta$.

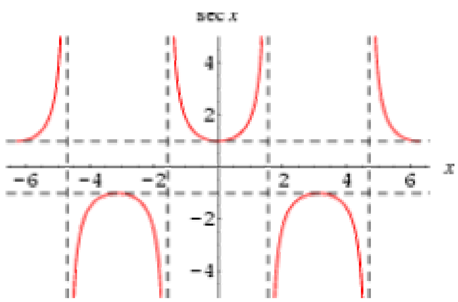


fig. 3.32

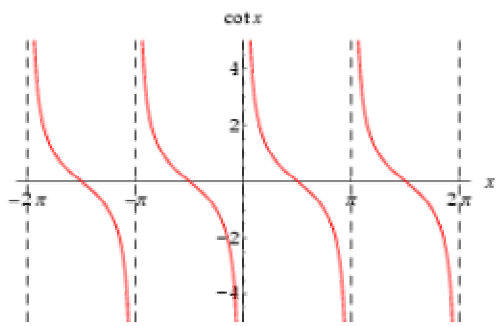


fig. 3.33

Two More Trigonometric Identities

In Chapter 2 we established the identity,

$$\sin^2\theta + \cos^2\theta = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textbf{(1)}$$

Dividing **(1)** throughout by $\cos^2\theta$ produces a new trigonometric identity:

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta} \quad \Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta \text{ or}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

Dividing **(1)** by $\sin^2\theta$ gives another identity,

$$1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta} \quad \Rightarrow \quad \boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

We now have two more tools at our disposal when solving trigonometric equations.

Example 3.10 Solve the equation $\tan^2 x + 2\sec x + 2 = 0$ in the range $0 \leq x \leq 2\pi$.

We notice that we can substitute for $\tan^2 x$ to get a quadratic in sec *x*.

Since $\tan^2 x = \sec^2 x - 1$ we can write:

$\sec^2 x - 1 + 2\sec x + 2 = 0 \quad \Rightarrow \quad \sec^2 x + 2\sec x + 1 = 0$.

We can now factorise:

$(\sec x + 1)(\sec x + 1) = 0$ or $(\sec x + 1)^2 = 0$

$$\Rightarrow \quad \sec x = -1 \quad \Rightarrow \quad \frac{1}{\cos x} = -1 \quad \Rightarrow \quad \cos x = -1$$

Hence we must have that $x = \pi$. This is the only solution in the given range.

Test 3.15 Solve the equation $\sec\theta = 3\operatorname{cosec}\theta$ for $0 \le \theta \le 2\pi$.

Test 3.16 Solve the equation $2\operatorname{cosec}^2\theta = 5 + 5\cot\theta$ in the interval $0 \le x \le \pi$.

3.3 Exponentials and Logarithms

The Function e^x and its Graph

Let us now introduce the number *e*. *e*, like π, is an irrational number. Moreover, it is a *transcendental* number (not the root of any ratioal polynomial). *e*, like π, is just a number. The value of *e* (to 75 decimal places) is shown below.

$e = 2.718281828459045235360287471352662497757247093699959574966967627724076630353...$

This number has some very special properties, which we will learn more about later.

e is often used as the base of logarithms. Recall from Chapter 2 the section 'Logarithms'. We noted that logarithms can have any base, but often we use logarithms with base 10. In fact, more often we use logarithms with base *e*. We use logarithms with base *e* so often that it has its own symbol, 'ln',

$\log_e x \equiv \ln x$. $\qquad$ $\ln x$ always means 'logarithm base *e* of *x*'.

The logarithmic function base *e* is often called 'the *natural* logarithm'.

One very important function in mathematics is the function $f(x) = e^x$. Recall from Chapter 2 the section 'The Graph of $y = a^x$'. In *fig. 2.15*, we plotted the graph of $y = a^x$ for different values of *a*. The graph of $f(x) = e^x$ is shown in *fig. 3.34*. The function $f(x) = e^x$ is known as '*the* exponential function'.

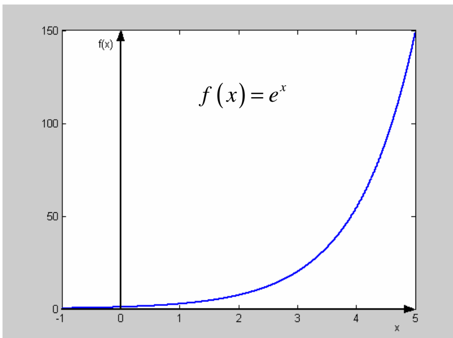


fig. 3.34

The Inverse of the Exponential Function

The inverse of the exponential function, e^x, is the logarithmic function base *e*, $\ln x$. Performing the exponential function on a number, and finding the natural logarithm of the result will take us back to the original number, i.e.

$\ln(e^x) = x$.

This demonstrates that if $f(x) = e^x$, then $f^{-1}(x) = \ln x$.

If $f(x) = e^x$, then $f^{-1}(x) = \ln x$

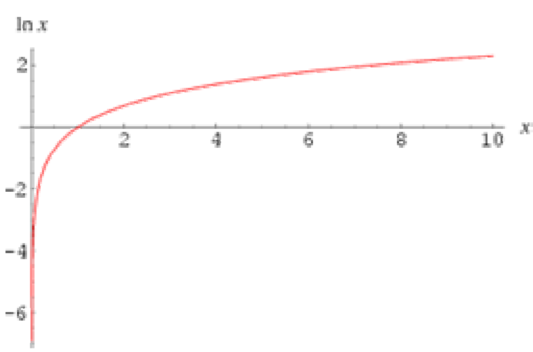


fig. 3.35

What does the graph of $y = \ln x$ look like? We have established that $\ln x$ is the inverse function of e^x. We know what the graph of e^x looks like (*fig. 3.34*), we know from the earlier section 'Inverse Functions and their Graphs' that the graph of the inverse function is a reflection of the graph of the original function in the line $y = x$. From *fig. 3.34*, we can plot the graph of $y = \ln x$ which is shown in *fig. 3.35*.

3.4 Differentiation

The Derivative of e^x

One of the most important features of the function $f(x)=e^x$ is that this function is its own derivative, i.e. if $f(x)=e^x$ then $f'(x)=e^x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)=e^x$$

This is the only (nontrivial) function that has this special property.

The Derivative of lnx

Before we look at the derivative of $\ln x$, we first mention an important point. We have introduced the symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a piece of notation. We have said that this is just a symbol, not a fraction. It is true, however, that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

Example 3.11 If $y=\ln x$, calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

In this example, we learn how to differentiate $y=\ln x$. First of all, form our knowledge of logarithms, we write,

$$y=\ln x \quad \Leftrightarrow \quad e^y=x.$$

So we have that $x=e^y$, now we can calculate $\frac{\mathrm{d}x}{\mathrm{d}y}$. We now know that the derivative of e^y is e^y.

$$x=e^y \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}y}=e^y.$$

Now, we know that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, so

$\frac{\mathrm{d}x}{\mathrm{d}y} = e^y \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y}$. The answer should be in terms of *x*. Remember that $x = e^y$, so

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$.

We now have the required result.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

The Derivative of sin*x*

A formal proof of the derivative of $\sin x$ is not given here, it can be found in A level maths text books.

Let us take a non-rigorous look at the problem. Look back at the graph of $\sin x$ (*fig. 2.3*). Remember that the derivative function is a function of how the gradient varies with *x*. It is clear that the derivative of $\sin x$ is a periodic function. It is also clear that the derivative of $\sin x$ has maximum (positive) values at $x = 0,\ \pi,\ 2\pi,\ ...$ since this is where $\sin x$ has its maximum positive gradients. We can see that the derivative function has minimum values at $x = -\pi,\ -2\pi,\ -3\pi,\ ...$ since this is where $\sin x$ has its minimum negative gradients.

We already know a function which has the required properties of the derivative function. This function is, of course, $\cos x$. Indeed, the derivative of $\sin x$ is $\cos x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

The Derivative of cosx

Again, we will not prove this result. A similar qualitative approach as above will suggest that $-\sin x$ fits the requirements of the derivative of $\cos x$. Indeed, $-\sin x$ is the derivative of $\cos x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

The Derivative of tanx

The result is simply stated here, however we will prove it later.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$

The Product Rule

So far we have only considered the derivatives of simple functions and linear combinations of these simple functions. We have not considered, for example, products of functions, like $x^2 \sin x$. There is an important rule for differentiating a product of two functions, called the product rule.

If f and g are both functions of x and $y = fg$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$

Let us prove this result. We have two functions $f(x)$ and $g(x)$ and we are considering the product $y = f(x)g(x)$. Suppose we change x by a small amount δx, which results in the function f changing by a small amount δf and the function g changing by a small amount δg and the function y changing by a small amount δy.

We have that $y = fg$. Changing x by a small amount δx will mean that,

$$y + \delta y = (f + \delta f)(g + \delta g)$$

$$\Rightarrow \quad \delta \mathrm{y} = fg + f\delta g + g\delta f + \delta f\delta g - y$$

But, $y = fg$, so,

$$\Rightarrow \quad \delta \mathrm{y} = fg + f\delta g + g\delta f + \delta f\delta g - fg$$

$$\Rightarrow \quad \delta \mathrm{y} = f\delta g + g\delta f + \delta f\delta g$$

Dividing through by δx gives,

$$\frac{\delta \mathrm{y}}{\delta x} = f\frac{\delta g}{\delta x} + g\frac{\delta f}{\delta x} + \frac{\delta f\delta g}{\delta x}$$

Now as the change in x tends to zero, the resulting changes in f, g, and y tend to zero, i.e. as $\delta x \to 0$, $\delta f \to 0$, $\delta f \to 0$ and $\delta g \to 0$, and $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta g}{\delta x} \to \frac{dg}{dx}$, $\frac{\delta f}{\delta x} \to \frac{df}{dx}$ and $\frac{\delta f \delta g}{\delta x} \to 0$.

Therefore, in the limit as x tends to zero, we have,

$$\frac{dy}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$ as required.

Example 3.12 If $y = x^2 \sin x$, calculate $\frac{dy}{dx}$.

We have a product of two functions, so use the product rule. We have the two functions, $f = x^2$ and $g = \sin x$. The product rule tells us to leave the first function alone and multiply by the derivative of the second function, we then add the second function left alone multiplied by the derivative of the first function.

Following this method gives:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(x^2\right)$$

$$= x^2 \cos x + \sin x(2x)$$

$$= x^2 \cos x + 2x \sin x$$

Example 3.13 If $y = \frac{e^x}{x}$, calculate $\frac{dy}{dx}$.

At first sight, this expression does not appear to be a product of two functions. We can write it as a product of two functions in the following way,

$$y = x^{-1}e^x.$$

Following the product rule we get,

$$\frac{dy}{dx} = x^{-1}\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^{-1}\right)$$

$$= x^{-1}e^x + e^x\left(-x^{-2}\right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$= \frac{e^x}{x}\left(1 - \frac{1}{x}\right)$$

Test 3.17 Use the product rule to evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+3\right)\left(3x^4-7x^2\right)\right)$

Test 3.18 Use the product rule to calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^2 x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x \sin x\right)$

Test 3.19 Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x \cos x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(2\cos x\right) = -1$. What does this tell you about the function $f(x) = \sin x \cos x - 2\cos x$?

The Quotient Rule

The quotient rule is a method for differentiating a quotient, or fraction, of the form $\frac{f(x)}{g(x)}$.

If $y = \frac{f(x)}{g(x)}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x).f'(x) - f(x).g'(x)}{\left(g(x)\right)^2}$

The quotient rule is not proved here. A proof can be found in A level pure maths text books.

Example 3.14 Given that $y = \frac{2x-3}{x^2+7}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Following the rule we have,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(x^2+7\right)\left(2x-3\right)' - \left(2x-3\right)\left(x^2+7\right)'}{\left(x^2+7\right)^2}$$

$$= \frac{(x^2+7).2-(2x-3).2x}{x^4+14x^2+49}$$

$$= \frac{2x^2+14-4x^2+6x}{x^4+14x^2+49}$$

$$= \frac{14+6x-2x^2}{x^4+14x^2+49}$$

Example 3.15 Given that $y = \frac{e^x}{\sin x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Following the rule for differentiating quotients, we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x.e^x - e^x(\cos x)}{\sin^2 x}$$

$$= \frac{e^x \sin x - e^x \cos x}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

> **Test 3.20** Use the quotient rule to evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-2x}{\cos x}\right)$

> **Test 3.21** Use the quotient rule to show that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$. (*Hint*: $\tan x = \frac{\sin x}{\cos x}$)

The Chain Rule

Introduction

This is an important rule used to differentiate more complicated functions that can be thought of as two functions in one, called **composite functions**. Let us look at an example of a composite function.

Consider the function $f(x) = (2x+3)^2$. If we wish to calculate the value of this function for $x = 2$, we split the calculation into two parts. First we calculate $2x+3$, for $x = 2$, this gives 7. Next we square the result, giving the final answer 49. The part of the function that we calculate first, $2x+3$ in this case, is called the **inside function**; the part of the function that we calculate second, the 'squared' part of the function in this case, is called the **outside function**. *NB*: The product rule would also work for this function.

$$f(x) = (2x+3)^2$$

Outside function

Inside function

Here is another example of a composite function, $f(x)=\sin(2x)$. What is the inside function and the outside function? If we were given a value to substitute in to this function, say $x=\frac{\pi}{2}$, we would first calculate $2x$ and then find the sine of the result. Therefore, the inside function is $2x$ and the outside function is $\sin(\text{inside function})$, the sine of the inside function.

Test 3.22 Identify the inside function and the outside function for the following composite functions:

a) $f_1(x)=\ln(2x+3)$ **b)** $f_2(x)=\dfrac{1}{(x^2-5)}$ **c)** $f_3(x)=e^{2x}$

d) $f_4(x)=\tan^2 x$

Method for Differentiating Composite Functions

When faced with the task of differentiating a composite function, we first need to identify the inside function and the outside function. Let us take the example $f(x)=(2x+3)^2$. We have established that the inside function is $2x+3$ and the outside function is $(\text{inside function})^2$. It may be useful to use a single symbol to stand for the inside function. Let us call the inside function *u*. i.e. let $u=2x+3$. We can now write that $f(x)=u^2$. The derivative of $f(x)$ with respect to *x* is given by:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\frac{\mathrm{d}f(x)}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$=2u\times\frac{\mathrm{d}u}{\mathrm{d}x}$$

Now since $u=2x+3$, $\frac{\mathrm{d}u}{\mathrm{d}x}=2$

So, $$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=2(2x+3)\times 2$$

$$=4(2x+3)$$

! Remember to give your final answer in terms of the original variable, *x*.

If $y=f(u)$ where *u* is a function of *x*, then

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$$

Example 3.16 Given that $y=\sin(3x+2)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The inside function here is $3x+2$. Let $u=3x+2$. We have $y=\sin u$. Now, from the chain rule, we have:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\cos u\times 3$$

$$=3\cos(3x+2)$$

Example 3.17 Given that $y=e^{2x}$, find $\frac{dy}{dx}$.

Here, the inside function is $2x$. Let $u=2x$. Then $y=e^{u}$. From the chain rule, we have that:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=e^{u}.2$$

$$=2e^{2x}$$

Example 3.18 Find $\frac{d}{dx}\left(\sin^2 x\right)$.

Let $y=\sin^2 x$. The inside function is $\sin x$. Let $u=\sin x$, so that $y=u^2$. From the chain rule, we have that:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=2u.\cos x$$

$$=2\sin x\cos x$$

Test 3.23 Using the chain rule, evaluate the following,

a) $\frac{d}{dx}\left(\sqrt{3x^2+x}\right)$ **b)** $\frac{d}{dx}\left(\cos^2 x\right)$ **c)** $\frac{d}{dx}\left(\ln\left(x^2\right)\right)$ **d)** $\frac{d}{dx}\left(\ln(2x)\right)$

3.5 Integration

Integration of e^x

What is the integral of e^x w.r.t. x? i.e. what is $\int e^x \mathrm{d}x$? We know that integration is the reverse process of differentiation, so we are asking the question, what function, when differentiated, gives e^x as the answer? We know that the answer to this question is e^x itself. So we have that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)=e^x \quad \Rightarrow \quad \int e^x \mathrm{d}x = e^x + c$$

Integration of *1/x*

What is $\int \frac{1}{x} \mathrm{d}x$? Again, the answer to the question is a function which, when differentiated gives $\frac{1}{x}$ as the answer. We know that the derivative of $\ln x$ is $\frac{1}{x}$, hence:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x} \quad \Rightarrow \quad \int \frac{1}{x} \mathrm{d}x = \ln x + c$$

Integration of sin*x* and cos*x*

Similarly, we have the following results:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x \quad \Rightarrow \quad \int \cos x \mathrm{d}x = \sin x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x \quad \Rightarrow \quad \int \sin x \mathrm{d}x = -\cos x + c$$

Test 3.24 Integrate the following function, $f(x)=\frac{1}{x}+x^2+\sin x-2\cos x+\frac{1}{2}e^x$.

Integrating Composite Functions (Functions of a Linear Function of *x*)

Earlier we developed a method of differentiating composite functions. This method tells us to differentiate the outside function and then multiply by the derivative of the inside function. There is an analogous method for integrating composite functions, which can be thought of as the reverse process of the chain rule.

Let us consider an example. Take the composite function $y=(2x-7)^4$. We know how to differentiate this function. We have that,

$\frac{\mathrm{d}y}{\mathrm{d}x}=8(2x-7)^3$ (check).

So, given the function $y=8(2x-7)^3$ how do we integrate it? We must reverse the differentiation process. To do this we integrate the outside function and divide by the derivative of the inside function.

So, if $f(x)=8(2x-7)^3$

inside function: $2x-7$. Let $2x-7=u$

Outside function: u^3 (we do not worry about the constant, 8)

So, $f(x)=8u^3$, $\int f(x)\,\mathrm{d}x=\frac{1}{\frac{\mathrm{d}u}{\mathrm{d}x}}\int 8(u(x))^3\,\mathrm{d}x$

$$=\frac{1}{2}.2.u^4+c$$

$$=(2x-7)^4+c$$

> To integrate a composite function, first identify the inside function and the outside function. Then we integrate the outside function and <u>divide</u> by the <u>derivative</u> of the inside function. But beware – **this method will only work when the *inside* function is *linear*.** Think about why this method does not work when the inside function is not linear – *Hint*: make a substitution (see next section) for the inside function, say *u*, and try to write down the integral solely in terms of the new variable, *u*.

! It may be possible to get confused when integrating composite functions because both integration and differentiation are used in the process. Remember to divide by the **derivative** of the inside function.

Example 3.19 Integrate the function $f(x)=\sqrt{3x-3}$.

The inside function is $3x-3$. Let $3x-3=u$. The outside function is $\sqrt{u}=u^{\frac{1}{2}}$.

So, $f(x)=u^{\frac{1}{2}}$. We integrate the outside function and divide by the derivative of the inside function.

$$\int f(x)\,\mathrm{d}x=\frac{1}{\frac{\mathrm{d}u}{\mathrm{d}x}}\int (u(x))^{\frac{1}{2}}\,\mathrm{d}x$$

$$=\frac{1}{3}\frac{2}{3}u(x)^{\frac{3}{2}}+c$$

$$=\frac{2}{9}(3x-3)^{\frac{3}{2}}+c=\frac{2}{9}\sqrt{(3x-3)^3}$$

Example 3.20 Evaluate $\int e^{3x+2}\,\mathrm{d}x$.

The inside function is $3x+2$. Let $3x+2=u$. The outside function is e^u.

Let $f(x)=e^{3x+2}=e^u$.

$$\int f(x)\,\mathrm{d}x=\frac{1}{\frac{\mathrm{d}u}{\mathrm{d}x}}\int e^u\,\mathrm{d}x=\frac{1}{3}e^u+c$$

$$=\frac{1}{3}e^{3x+2}+c$$

Test 3.25 Evaluate $\int 2(x-5)^4\,\mathrm{d}x$

Test 3.26 Evaluate $\int \cos(3x)\,\mathrm{d}x$

Test 3.27 A function, $f(x)$, is differentiated to give, $f'(x)=\frac{1}{x+2}-\sin(3x)$. Suggest a formula for $f(x)$

Integration by Substitution

When faced with the task of integrating a product, like $x(x+4)^3$, you should consider the method of integration by substitution. This method is perhaps best explained by considering a step-by-step example.
Suppose we wish to evaluate $\int x(x+4)^3 \, \mathrm{d}x$. As the name of the method suggests, we proceed by making an algebraic substitution. In this example, we let $u = x+4$. It will become clear why we chose this particular substitution as we proceed through the example. The aim of the game is to replace every expression involving *x* in the original problem with an expression involving *u*.

Now, we have decided that $u = x+4$. Rearranging this we can see that $x = u-4$, so $u-4$ replaces the occurrence of x in the original problem. We also have a 'd*x*' in the original problem. Now, since $u = x+4$, we can say that $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \Rightarrow \quad \mathrm{d}x = \frac{\mathrm{d}u}{1} = \mathrm{d}u$. So d*x* is replaced by d*u*.

The original problem was $\int x(x+4)^3 \, \mathrm{d}x$. Making the substitution leads to the following form of the problem:

$$\int (u-4)u^3 \, \mathrm{d}u \, .$$

Multiplying out the brackets gives:

$$\int u^4 - 4u^3 \, \mathrm{d}u \qquad \text{which can easily be evaluated.}$$

$$\int u^4 - 4u^3 \, \mathrm{d}u = \frac{1}{5}u^5 - u^4 + c \, .$$

Remember that the original question was posed in terms of *x*, so the final answer must also be stated in terms of *x*. So we have:

$$\int x(x+4)^3 \, \mathrm{d}x = \frac{1}{5}(x+4)^5 - (x+4)^4 + c \, .$$

Note: In this example, we could have multiplied out the original integrand $x(x+4)^3$. If the integrand had been, for example $x(x+4)^7$, however, it is clear that multiplying out the bracket is not sensible.

Example 3.21 Evaluate $\int x\sqrt{2x-3} \, \mathrm{d}x$ by using the substitution $u = 2x-3$.

Sometimes, as in this problem, we are given an appropriate substitution. Other times, we have to think for ourselves.

If $u = 2x - 3$, then $x = \dfrac{u+3}{2}$. Also, we have that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \Rightarrow \quad \mathrm{d}x = \dfrac{\mathrm{d}u}{2}$.

So, $\int x\sqrt{2x-3}\ \mathrm{d}x$ becomes $\int \dfrac{u+3}{2}.u^{\frac{1}{2}}\ \dfrac{\mathrm{d}u}{2} = \dfrac{1}{4}\int u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\ \mathrm{d}u$

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}}\right) + c$$

$$= \frac{1}{10}\sqrt{u^5} + \frac{1}{2}\sqrt{u^3} + c$$

$$= \frac{1}{10}\sqrt{(2x-3)^5} + \frac{1}{2}\sqrt{(2x-3)^3} + c\,.$$

Example 3.22 Evaluate $\int (x-1)(2x+5)^4\ \mathrm{d}x$.

Often, we substitute for a term that is raised to a power, in this case the term $2x+5$. Let $u = 2x+5$. Now, we need to express $x-1$ in terms of u.

Since $u = 2x+5$ we have that $x = \dfrac{u-5}{2}$, so $x - 1 = \dfrac{u-5}{2} - 1 = \dfrac{u-7}{2}$.

Now $u = 2x+5$, so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \Rightarrow \quad \mathrm{d}x = \dfrac{\mathrm{d}u}{2}$.

So, $\int (x-1)(2x+5)^4\ \mathrm{d}x$ becomes $\int \dfrac{u-7}{2}u^4\,\dfrac{\mathrm{d}u}{2} = \dfrac{1}{4}\int u^5 - 7u^4\ \mathrm{d}u$

$$= \frac{1}{4}\left(\frac{1}{6}u^6 - \frac{7}{5}u^5\right) + c$$

$$= \frac{1}{4}\left[\frac{1}{6}(2x+5)^6 - \frac{7}{5}(2x+5)^5\right] + c$$

Example 3.23 Evaluate $\int 2x^3\left(\frac{1}{2}x^4 - 7\right)^3 \mathrm{d}x$.

Here we make the substitution $u = \frac{1}{2}x^4 - 7 \quad \Rightarrow \quad \frac{du}{dx} = 2x^3 \quad \Rightarrow \quad dx = \frac{du}{2x^3}$.

So, $\int 2x^3\left(\frac{1}{2}x^4 - 7\right)^3 dx$ becomes $\int 2x^3.u^3 \frac{du}{2x^3}$. We notice that the $2x^3$ cancels to give,

$$\int u^3\, du = \frac{1}{4}u^4 + c = \frac{1}{4}\left(\frac{1}{2}x^4 - 7\right)^4 + c.$$

Notice that in this example, the problem was made easy because the $2x^3$ cancelled. This is because $2x^3$ is the derivative of $\frac{1}{2}x^4 - 7$.

Test 3.28 Integrate the function $f(x) = x(3x-1)^2$

Test 3.29 Evaluate $\int (2x-1)(x-1)^6\, dx$ using the substitution $u = x - 1$

Test 3.30 Evaluate $\int 4x^3(x-7)^3\, dx$

The Integral of *f′(x)/x*

The integral of a function of the form $\frac{f'(x)}{f(x)}$ is the natural logarithm of the function on the denominator, $f(x)$. i.e.

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

We can illustrate why this is true by using the substitution $u = f(x)$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x)$ and so $\mathrm{d}x = \frac{\mathrm{d}u}{f'(x)}$. Then the integral becomes,

$$\int \frac{f'(x)}{u}\frac{\mathrm{d}u}{f'(x)} = \int \frac{1}{u}\mathrm{d}u = \ln|u| + c = \ln|f(x)| + c.$$

Example 3.24 Evaluate $\int \frac{18x+12}{(3x+2)^2}\mathrm{d}x$.

We can write down the answer to this problem straight away by noticing that the derivative of the denominator is $\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+2)^2\right) = 6(3x+2) = 18x+12$, which is exactly the numerator. So we have the answer $\int \frac{18x+12}{(3x+2)^2}\mathrm{d}x = \ln(3x+2)^2$.

Example 3.25 Evaluate $\int \frac{x}{1+x^2}\mathrm{d}x$.

Now, in this case the numerator is not exactly the derivative of the denominator, but it is $\frac{1}{2}$ times the derivative of the denominator. This integral is of the form

$\int \frac{\frac{1}{2}f'(x)}{f(x)}\mathrm{d}x = \frac{1}{2}\int \frac{f'(x)}{f(x)}\mathrm{d}x = \frac{1}{2}\ln|f(x)| + c$. So the answer to the question is,

$$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln\left(1+x^2\right).$$

Test 3.31 Evaluate $\int \frac{3x}{x^3-6}\mathrm{d}x$

Test 3.32 Evaluate $\int \frac{\frac{1}{2x} - \sin x\cos x}{\ln x - \sin^2 x}\mathrm{d}x$

Integration by Parts

This is another method of integrating a product of functions. We have studied the product rule, which we use to differentiate products of functions, the method of integration by parts can be derived from the product rule for differentiation.

Recall that, $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$ ....(†)

where u and v are both functions of x. Integrating both sides of (†) with respect to x gives:

$$uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$$

$$\Rightarrow \quad \int uv'\ \mathrm{d}x = uv - \int vu'\ \mathrm{d}x$$

This is the method of integration by parts. It is a little difficult to see how it works from this formula, so let us consider an example.

Example 3.26 Calculate $\int xe^x\ \mathrm{d}x$.

Here we set $u = x$, $v' = e^x$. So we have $u' = 1$, $v' = e^x$ and $v = e^x$.

We have that, $\int uv'\ \mathrm{d}x = uv - \int vu'\ \mathrm{d}x$

$$\Rightarrow \quad \int xe^x\ \mathrm{d}x = xe^x - \int e^x.1\ \mathrm{d}x$$

$$= xe^x - e^x + c$$

$$= e^x(x-1) + c$$

The important thing to remember about this method is that only one of the functions, u or v, has to be integrated, the other is differentiated. We can choose which function we integrate and which function we differentiate. In this example, we choose $u = x$ so that when we differentiate it, we get $u' = 1$, a constant, which makes the problem very easy. If we had made the wrong decision and set $u = e^x$, it would not have been so easy to solve the problem.

Example 3.27 Evaluate $\int x\cos x\ \mathrm{d}x$.

Here we choose $u = x$ and $v' = \cos x$. It is not necessary to write down your choices for u and v every time you use this method, nor is it necessary to write down the integration by parts formula once you are comfortable with the method.

$$\int x\cos x \;\mathrm{d}x = x.\sin x - \int \sin x.1 \;\mathrm{d}x$$

$$= x\sin x + \cos x + c\,.$$

Example 3.28 Evaluate $\int x^2 \sin x \;\mathrm{d}x$.

In the previous two examples, we had a single x term which we chose to be u so that when we differentiated it we got a constant term. In this example, we have an x^2 term. When we differentiate this term once we get $2x$. Notice, however, that if we differentiate it a second time we do get a constant, 2. This suggests that we need to use the method of integration by parts twice in this example.

Let us choose $u = x^2$ and $v = \sin x$. Working through the method we get,

$$\int x^2 \sin x \;\mathrm{d}x = -x^2 \cos x - \int (-\cos x).2x \;\mathrm{d}x$$

$$= -x^2 \cos x + \int 2x.\cos x \;\mathrm{d}x$$

It may seem at this stage that we have not made any progress, since we still cannot integrate $2x.\cos x$. We can however use integration by parts a second time to integrate this product. Let us rewrite the last line of the calculation as follows,

$$\int x^2 \sin x \;\mathrm{d}x = -x^2 \cos x + \int 2x.\cos x \;\mathrm{d}x$$

$$= -x^2 \cos x + I$$

Now, $I = \int 2x.\cos x \;\mathrm{d}x$

$$= 2x\sin x - \int \sin x.2 \;\mathrm{d}x$$

$$= 2x\sin x + 2\cos x + c$$

So, the answer to the original problem is,

$$\int x^2 \sin x \;\mathrm{d}x = -x^2 \cos x + 2x\sin x + 2\cos x + c$$

(which you may like to write a little neater)

Example 3.29 Evaluate $I = \int e^x \sin x \, dx$.

Notice that in this example, neither of the terms will ever reduce to a constant, no matter how many times we differentiate it. We can still solve this problem by using integration by parts twice. It does not matter in this case which term we choose to differentiate and which term we choose to integrate. Let us choose to differentiate the e^x term and integrate the $\sin x$ term.

$$\int e^x \sin x \, dx = -\cos x . e^x - \int (-\cos x) . e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx .$$

Now let us use integration by parts a second time to evaluate $\int e^x \cos x \, dx$. Again we will chose to differentiate the e^x term and integrate the $\cos x$ term (although in this case it will also work the other war around).

$$\int e^x \cos x \, dx = e^x \sin x - \int \sin x . e^x \, dx$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x - I \qquad \text{(Recall that } I = \int e^x \sin x \, dx\text{)}$$

Substituting this result into the original problem gives,

$$I = \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - I$$

Some simple algebraic manipulation gives:

$$2I = -e^x \cos x + e^x \sin x \quad \Rightarrow \quad I = \frac{1}{2} e^x (\sin x - \cos x)$$

i.e. $I = \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$

Test 3.33 Evaluate $\int x \sin x \, dx$

Test 3.34 By using integration by parts, evaluate $\int x (x-2)^4 \, dx$

Test 3.35 Evaluate $\int x^2 e^x \, \mathrm{d}x$

Test 3.36 By using integration by parts, evaluate $\int \ln x \, \mathrm{d}x$. *Hint*: Think of $\ln x$ as $1 \times \ln x$. Choose $u = \ln x$, $v = 1$

Two More Standard Integrals

Here we will look at how to integrate the expressions $\frac{1}{a^2+x^2}$ and $\frac{1}{\sqrt{a^2-x^2}}$, where a is a constant. The method that we use to solve these problems is substitution, but the choice of substitution is not obvious.

1) $\int \frac{1}{a^2+x^2} \, \mathrm{d}x$. To solve this problem, we use the substitution $x = a\tan\theta$. This may seem like a strange substitution to make, but we will see how it works as we proceed through the problem.

Let $x = a\tan\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec^2\theta \quad \Rightarrow \quad \mathrm{d}x = a\sec^2\theta \, \mathrm{d}\theta$.

So, $I = \int \frac{1}{a^2+x^2} \, \mathrm{d}x$ becomes $I = \int \frac{1}{a^2+(a\tan\theta)^2} . a\sec^2\theta \, \mathrm{d}\theta$.

Now, since $1+\tan^2\theta = \sec^2\theta$, we can write this as,

$$I = \int \frac{1}{a^2\sec^2\theta} . a\sec^2\theta \, \mathrm{d}\theta$$

$$= \int \frac{1}{a} \, \mathrm{d}\theta$$

$$\frac{\theta}{a} + c$$

Now, since $x = a\tan\theta$, $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.

So, $I = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

So we have the result,

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

This is a standard result. If you recognise an integrand to be of the form $\frac{1}{a^2 + x^2}$, for example $\frac{1}{36 + x^2}$ or $\frac{1}{x^2 + 9}$, you can write down the answer without any calculation.

2) $\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$. To solve this problem, we use the substitution $x = a\sin\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta \quad \Rightarrow \quad \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$.

So, $I = \int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$ becomes $I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}}.a\cos\theta\,\mathrm{d}\theta$

$$= \int \frac{1}{\sqrt{a^2\left(1 - \sin^2\theta\right)}}.a\cos\theta\,\mathrm{d}\theta.$$

Now, since $1 - \sin^2\theta \equiv \cos^2\theta$, we can write this as,

$$I = \int \frac{1}{a\cos\theta}.a\cos\theta\,\mathrm{d}\theta$$

$$= \int 1\,\mathrm{d}\theta = \theta + c.$$

Now, since $x = a\sin\theta$, $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Hence,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Example 3.30 Calculate $\int \frac{1}{x^2 + 16}\,\mathrm{d}x$.

This is of the standard form $\int \frac{1}{a^2+x^2}\,\mathrm{d}x$ with $a=4$.

Hence, we can write down the answer straight away, $\int \frac{1}{x^2+16}\,\mathrm{d}x = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$.

Example 3.31 Calculate $\int \frac{7}{64+4x^2}\,\mathrm{d}x$.

We can write this in the form $7\int \frac{1}{8^2+(2x)^2}\,\mathrm{d}x$. This is now in the standard form

$\int \frac{1}{a^2+x^2}\,\mathrm{d}x$ with $a=8$ and $x=2x$.

Now we can write down the answer straight away, $\int \frac{7}{64+4x^2}\,\mathrm{d}x = \frac{7}{8}\tan^{-1}\left(\frac{2x}{8}\right)$

$$= \frac{7}{8}\tan^{-1}\left(\frac{x}{4}\right).$$

Test 3.37 Calculate each of the following integrals,

a) $\int \frac{1}{4+x^2}\,\mathrm{d}x$ **b)** $\int \frac{3}{\sqrt{9-x^2}}\,\mathrm{d}x$ **c)** $\int \frac{2}{\sqrt{25-100x^2}}\,\mathrm{d}x$

Test 3.38 Evaluate $\int \frac{2x-1}{x^2+16}\,\mathrm{d}x$

Volume of Revolution

Look at the graph of $y=x^2$ in *fig. 3.34*.

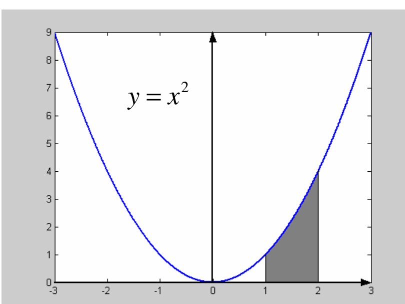


fig. 3.34

We have seen how to calculate the value of the shaded area using integration. Now, imagine that the graph is in 3 dimensional space and the whole parabola moves through a full turn (2π radians) about the x-axis. Imagine the solid that the shaded area would sweep out. Make a sketch of the 3 dimensional solid in the box below (*fig. 3.35*).

In this section, we will learn how to calculate the area of this solid and other similar *volumes of revolution*.

The way we go about solving this problem is to imagine the solid in *fig. 3.35* sliced up vertically into a number of very thin slices each of width δx. The radii of the discs will vary as we move along the solid, the radius of each disc will be $y = x^2$, which depend on where we are in relation to the *y* axis.

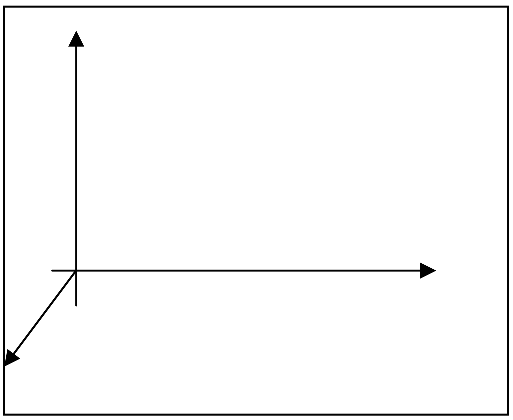


fig. 3.35

Note: technically the slices of the solid will not be perfect discs ass shown in *fig. 3.36*, but the thinner the slices, the closer the pieces will be to perfect discs.

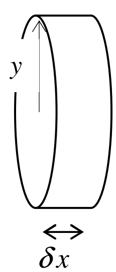


fig. 3.36

The volume of the disc opposite is $\pi y^2 \delta x$. As we have mentioned, each disc will have a different radius, *y*, which varies with *x*. To find the total volume of the solid in *fig. 3.35*, we add up the volumes of the individual thin discs from $x = 1$ to $x = 2$ to give the total volume of the solid as,

$$V = \sum_{x=1}^{2} \pi y^2 \delta x .$$

As we make δx smaller and smaller, the calculated volume gets closer and closer to the true volume of the solid, which is given by,

$$V = \pi \int_1^2 y^2 \; \mathrm{d}x .$$

To state a general result:

The volume, *V*, of the solid of revolution created when the area under a graph $y = f(x)$ from $x = a$ to $x = b$ is rotated through 2π radians about the *x*-axis is given by:

$$V = \pi \int_a^b y^2 \, dx$$

Example 3.32 Find the volume of revolution formed when the area under the graph of $y = 3x^2 + 4$ from $x = 1$ to $x = 5$ is rotated through 2π radians about the *x*-axis.

It is often helpful to make a sketch, especially for more complicated questions. The required volume is given by:

$$V = \pi \int_1^5 y^2 \, dx .$$

Now, $y = 3x^2 + 4$, so

$$y^2 = \left(3x^2 + 4\right)^2 = 9x^4 + 24x^2 + 16$$

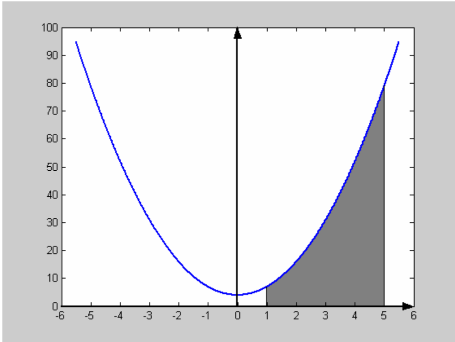


fig. 3.37

So, $V = \pi \int_1^5 9x^4 + 24x^2 + 16 \, dx$

$$= \pi \left[\frac{9}{5} x^5 + 8x^3 + 16x \right]_1^5$$

$$= \frac{33396}{5} \pi \ \left(\text{units}^3\right)$$

Example 3.33 Find the volume of revolution formed when the area bounded by the graphs $y = x^2 + 3$ and $y = -x^2 + 5$ is rotated through 2π radians about the *x*-axis.

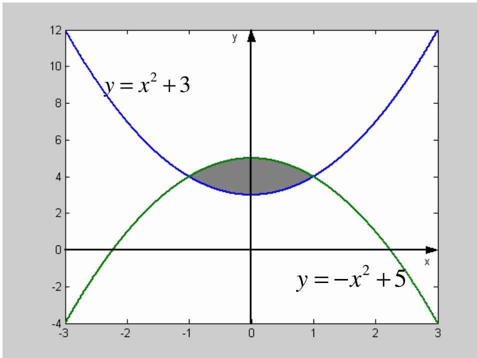


fig. 3.38

The area in this question is shown in *fig. 3.38*. To find the limits of integration, we need to work out where the two graphs intersect. To do this we solve the equation $x^2 + 3 = -x^2 + 5$, which gives $x = \pm 1$ (check). The required volume is given by:

$$V = \pi \int_{-1}^{1} \left(-x^2 + 5\right)^2 \, dx - \pi \int_{-1}^{1} \left(x^2 + 3\right)^2 \, dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 25x\right]_{-1}^{1} - \pi \left[\frac{1}{5}x^5 + 2x^2 + 9x\right]_{-1}^{1}$$

$$= \pi \frac{656}{15} - \pi \frac{112}{5}$$

$$= \pi \frac{320}{15} = \pi \frac{64}{3} \quad \left(\text{units}^3\right)$$

So far we have looked at volumes of revolution about the *x*-axis only. It may also be necessary for us to calculate a volume of revolution about the *y*-axis. The method is very similar to the problems of revolution about the *x*-axis that we have solved so far.

The volume, *V*, of the solid of revolution created when the area under a graph $y = f(x)$ from $x = a$ to $x = b$ is rotated through 2π radians about the *y*-axis is given by:

$$V = \pi \int_{a}^{b} x^2 \, dy$$

Example 3.34 Find the volume of revolution created when the area bounded by the curve $y = 2x^2$ and the lines $x = 3$ to $x = 5$ and the *x*-axis is rotated through 2π radians about the *y*-axis.

The required volume, V, is given by,

$$V = \pi \int_{x=3}^{x=5} x^2 \; dy \; .$$

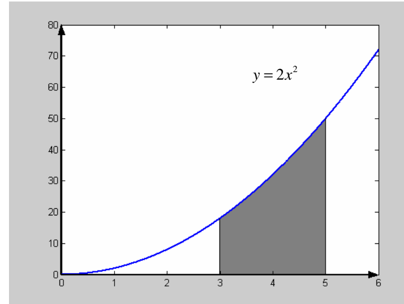


fig. 3.39

Now, since we are integrating with respect to y, we must have the integrand (and limits) in terms of y. We do this by using the relation $y = 2x^2 \;\Rightarrow\; x^2 = \frac{y}{2}$. So,

$V = \pi \int_{x=3}^{x=5} x^2 \; dy$ becomes,

$$V = \pi \int_{y=18}^{y=50} \frac{y}{2} \; dy = \pi \left[\frac{y^2}{4} \right]_{18}^{50} = 544\pi \; .$$

! Remember, x^2 appears in the integral for volumes of revolution about the y-axis; y^2 appears in the integral for volumes of revolution about the x-axis.

Test 3.39 Calculate the volume of revolution created when the area bounded by the curves $y = x^2$ and $y = 6x - 8$ is rotated through 2π radians about the x-axis.

Test 3.40 Calculate the volume of revolution created when the area below the curve $y = x^3$ from $x = 1$ to $x = 6$ is rotated through 2π radians about the y-axis.

Test 3.41 Calculate the volume of revolution formed when the area below the curve $y = -x^2 + 7$ from $x = 0$ to $x = 2$ is rotated through π radians about the x-axis.

3.6 Numerical Methods

Introduction and Interval Bisection

In real life situations, we are often faced with equations which have no analytic solution. That is to say we cannot find an *exact* solution to the equation. For example, we can solve the equation $x^2 + x - 2 = 0$ by factorising $(x+2)(x-1) = 0 \;\Rightarrow\; x = -2$ or $x = 1$. We have successfully solved the equation analytically to find the *exact* solutions. What about the equation $\cos x - x = 0$. Can you solve this equation? Well, unfortunately this equation can not be solved analytically unlike the previous example. We can not find the *exact* solution of this equation using algebraic, or any other techniques. So, do we have any hope of solving this equation? Well, we can find the *approximate* solution or solutions to the equation $\cos x - x = 0$. In fact, we can find the solution or solutions to an arbitrary degree of accuracy, however the more accurate we require our solution(s), the longer the process.

So, suppose we want to solve the equation $\cos x - x = 0$. The graph of $y = f(x) = \cos x - x$ is shown in *fig. 3.40*. We know that the solution of $f(x) = 0$ corresponds to the point where the graph of $f(x)$ cuts the *x*-axis. So we can tell, just from plotting the graph, that the solution is somewhere around $x = 0.7$. We notice that to the left of the root, the function is positive and to the right of the root the function is negative.

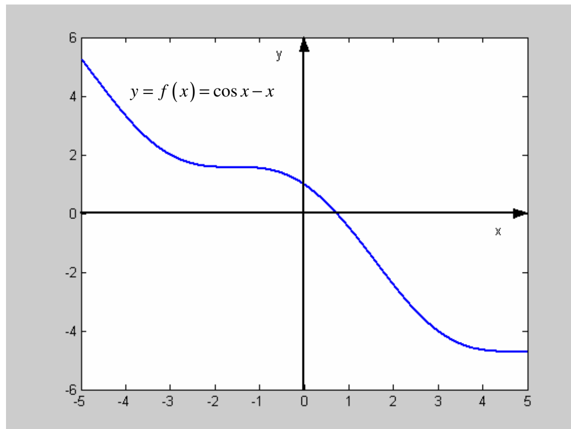


fig. 3.40

In general, the sign of a function, $f(x)$, to the left of a root is opposite to the sign of the function to the right of thee root. We can use this simple fact to help us find the roots of equations. For example to solve $\cos x - x = 0$, we can calculate the value of the function at a few points and see if we get a change of sign: $f(0) = \cos(0) - 0 = 1 - 0 = 1$ (positive), $f(1) = \cos(1) - 1 \approx -0.4597$ (negative), so we can say that there is a zero somewhere between $x = 0$ and $x = 1$. To get a more accurate approximation to the root, we could look at the value of the function $f(x)$ at the point mid-way between $x = 0$ and $x = 1$, i.e. at the point $x = 0.5$. We see that $f(0.5) = \cos(0.5) - 0.5 \approx 0.3776$ (positive). So now we can say that the root lies somewhere between $x = 0.5$ and $x = 1$. To get a better approximation, we look at the value of $f(x)$ at the point mid-way between $x = 0.5$ and $x = 1$, i.e. at the point $x = 0.75$. We see that $f(0.75) = \cos(0.75) - 0.75 \approx -0.0183$ (negative). So now we can say that the root lies somewhere between $x = 0.5$ and $x = 0.75$. We can continue in this way to give the following,

$f(0.5) \approx 0.3776 > 0$	$f(0.75) \approx -0.0183 < 0$	mid point = 0.626
$f(0.625) \approx 0.1860 > 0$	$f(0.75) \approx -0.0183 < 0$	mid point = 0.6875
$f(0.6875) \approx 0.0853 > 0$	$f(0.75) \approx -0.0183 < 0$	mid point = 0.71875

$f(0.71875) \approx 0.0339 > 0$ $\quad f(0.75) \approx -0.0183 < 0$ $\quad$ mid point = 0.73438

$f(0.73438) \approx 0.0079 > 0$ $\quad f(0.75) \approx -0.0183 < 0$ $\quad$ mid point = 0.74219

$f(0.74219) \approx -0.0052 < 0$ $\quad f(0.73438) \approx 0.0079 > 0$ $\quad$ mid point = 0.73829

At this stage, we can say that the root of the equation $\cos x - x = 0$ lies between $x = 0.73$ and $x = 0.74$. So, at this stage we can say that the root of the equation is 0.7 to one decimal place. We could continue to achieve better accuracy.

NB: The equation $\cos x = 0$ has an infinite number of roots. Does the equation $\cos x - x = 0$ have only the one root we have approximated, or are there other roots outside the domain we have considered? By plotting the graphs of $y = x$ and $y = \cos x$ we can see that they cross only once, therefore there is only one root of $\cos x = x$.

Example 3.35 Find the positive root of $10^x = x + 5$ to one decimal place.

We are told to find the positive root to the equation, so we need not bother with calculating the value of the function for negative *x*. Let us calculate $f(0), f(1), f(2), f(3), \ldots$ until we find a sign change.

$f(0) = 10^0 - 0 - 5 = -4 < 0$, $f(1) = 10 - 1 - 5 = 4 > 0$, so we can say that the root lies between $x = 0$ and $x = 1$. Now we consider the function at $x = 0.5$.

$f(0.5) = -2.34 < 0$, so we can say that the root lies between $x = 1$ and $x = 0.5$. Now we consider the function at $x = 0.75$.

$f(0.75) = -0.127 < 0$, so we can say that the root lies between $x = 1$ and $x = 0.75$. Now we consider the function at $x = 0.875$.

$f(0.875) = 1.623 > 0$, so we can say that the root lies between $x = 0.75$ and $x = 0.875$. Now we consider the function at $x = 0.8125$.

$f(0.8125) = 0.6813 > 0$, so we can say that the root lies between $x = 0.75$ and $x = 0.8125$. Now we consider the function at $x = 0.7813$.

$f(0.7813) = 0.2624 > 0$, so we can say that the root lies between $x = 0.75$ and $x = 0.7813$. Now we consider the function at $x = 0.7657$.

$f(0.7657) = 0.0647 > 0$, so we can say that the root lies between $x = 0.75$ and $x = 0.7657$. Now we consider the function at $x = 0.7579$.

$f(0.7579) = -0.0313 < 0$, so we can say that the root lies between $x = 0.7657$ and $x = 0.7579$.

At this stage, we can say that the root of $10^x = x + 5$ is 0.8 to one decimal place.

Test 3.42 Find the positive root of $x^4 - 2x^3 - 1 = 0$ to two decimal places.

Fixed Point Iteration

To solve the equation $f(x) = 0$, we may rearrange this into the form $x = \phi(x)$ so that if x satisfies $f(x) = 0$, then it also satisfies $x = \phi(x)$. Then the fixed point iteration is the computation $x_{n+1} = \phi(x_n)$. This sounds rather complicated, so let us consider an example.

Example 3.36 Using the fixed point iteration method, solve the equation $\cos x - x = 0$.

The first step is to rearrange the equation $\cos x - x = 0$ into the form $x =$ some function of x. The most obvious way to do this is to arrange it into the form $x = \cos x$. Now, the iteration formula is $x_{n+1} = \cos x_n$. We start with an initial guess to the root, x_0. Let us make our initial guess $x_0 = 0.7$. We then feed the initial guess into the iteration formula, to produce a better approximation of the solution, x_1. We then feed x_1 into the iteration formula to produce a better approximation, x_2.

So, $x_1 = \cos x_0$ with initial guess $x_0 = 0.7$, we get $x_1 = \cos(0.7) = 0.7648$.

$x_2 = \cos x_1 = \cos(0.7648) = 0.7215$

$x_3 = \cos x_2 = \cos(0.7215) = 0.7508$

$x_4 = \cos x_3 = \cos(0.7508) = 0.7311$

$x_4 = \cos x_3 = \cos(0.7508) = 0.7311$

$x_5 = \cos x_4 = \cos(0.7311) = 0.7444$

$x_6 = \cos x_5 = \cos(0.7344) = 0.7422$

$x_8 = \cos x_7 = \cos(0.7370) = 0.7405$

$x_9 = \cos x_8 = \cos(0.7405) = 0.7381$

$x_{10} = \cos x_9 = \cos(0.7381) = 0.7397$

$x_{11} = \cos x_{10} = \cos(0.7397) = 0.7387$

$x_{12} = \cos x_{11} = \cos(0.7387) = 0.7394$

$x_7 = \cos x_6 = \cos(0.7422) = 0.7370$

We can see that after 12 iterations, the approximation is settling down to a number around 0.739.

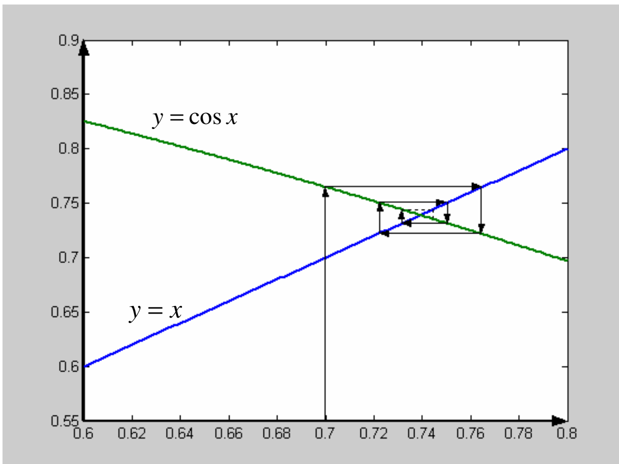


fig. 3.41

fig. 3.41 illustrates the iteration process. We can see in this case that the iterations produce better and better approximations each time – we can see this in the diagram because the arrows are getting closer and closer to the required root. The diagram in *fig. 3.41* is sometimes called a **cobweb diagram**. When the iterations get closer and closer to the required root, we say that the iteration **converges**.

Iterations do not always converge, as we will see later.

Often, there is more than one way to rearrange a given equation into the form $x = \text{some function of } x$. Depending on how we do this, the iteration may or may not converge. Some rearrangements lead to iterations that converge much faster than others.

Suppose, in example 3.36, we decided to rearrange the equation $\cos x - x = 0$ as $x = \frac{1}{2}x + \frac{1}{2}\cos x$ (check that this is a correct rearrangement). Performing the iteration $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}\cos x_n$ with initial guess $x_0 = 0.7$ gives:

$$x_1 = \frac{1}{2}\times 0.7 + \frac{1}{2}\cos(0.7) = 0.7324$$

$$x_2 = \frac{1}{2}\times 0.7324 + \frac{1}{2}\cos(0.7324) = 0.7379$$

$$x_3 = \frac{1}{2}\times 0.7379 + \frac{1}{2}\cos(0.7379) = 0.7389$$

$$x_4 = \frac{1}{2}\times 0.7389 + \frac{1}{2}\cos(0.7389) = 0.7390$$

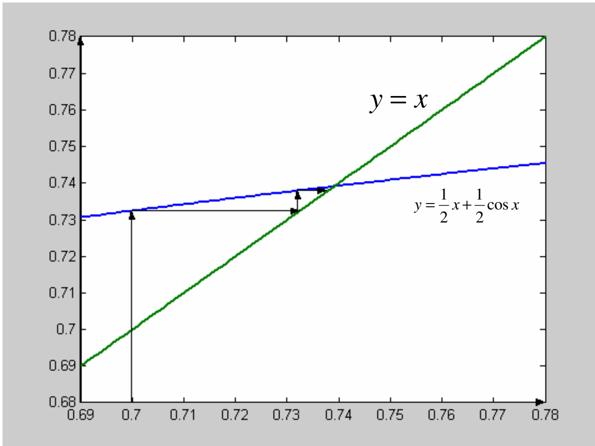


fig. 3.42

This time we can see that after just 4 iterations, the approximation is settling down to a number around 0.739. So this particular rearrangement leads to a process which converges much faster than before.

fig. 3.42 illustrates the iteration process. We can see that this rearrangement leads to a much faster convergence. Illustrations of this kind are sometimes called **staircase diagrams**.

Finally consider the equation $\cos x - x = 0$ rearranged as $x = \frac{3}{2}x - \frac{1}{2}\cos x$ (check that this is a correct rearrangement). Performing the iteration $x_{n+1} = \frac{3}{2}x_n - \frac{1}{2}\cos x_n$ with initial guess $x_0 = 0.7$ gives:

$$x_1 = \frac{3}{2}\times 0.7 - \frac{1}{2}\cos(0.7) = 0.6676$$

$$x_1 = \frac{3}{2}\times 0.6676 - \frac{1}{2}\cos(0.6676) = 0.6087$$

$$x_2 = \frac{3}{2}\times 0.6087 - \frac{1}{2}\cos(0.6087) = 0.5029$$

$$x_3 = \frac{3}{2}\times 0.5029 - \frac{1}{2}\cos(0.5029) = 0.3162$$

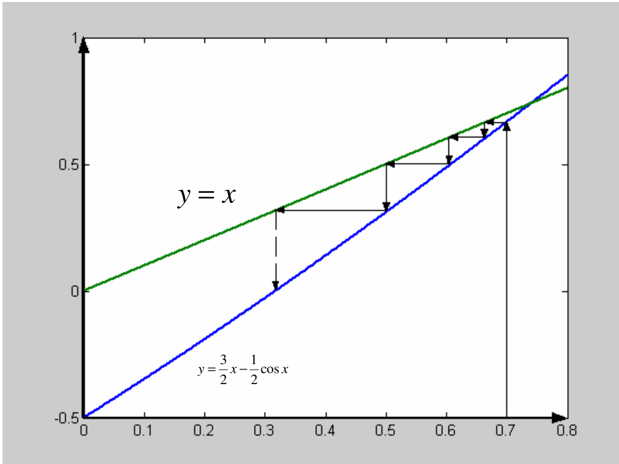


fig. 3.42

In this case, the iteration does not take us closer and closer to the desired root. In fact, it takes further and further away from the desired root. This is illustrated in *fig. 3.43*. In cases like this, we say that the iteration **diverges**.

Test 3.43 Use a suitable iteration to find the positive root of the equation $x^2 + \sin x = 1$ correct to 2 decimal places.

The Mid Point Rule

This is a method of finding the approximate value of the integral of a function when the function in question can not be integrated analytically. The general method is to approximate the area under the graph of $f(x)$ by splitting the area up into simple shapes (rectangles) that we can easily find the area of.

Let us consider a simple example. Suppose we want to find the area under the graph of $f(x) = x^2$ between the points $x = 0$ and $x = 5$ (of course, we can actually find this area analytically, it is $\int_0^5 x^2 \mathrm{d}x = 41.67$ (check)). One way of proceeding would be to take the mid point of the limits of integration ($\frac{0+5}{2} = 2.5$) and use the area of the rectangle illustrated in *fig. 3.43*, with width 5 and height $f(2.5) = 6.25$, to approximate $\int_0^5 x^2 \mathrm{d}x$.

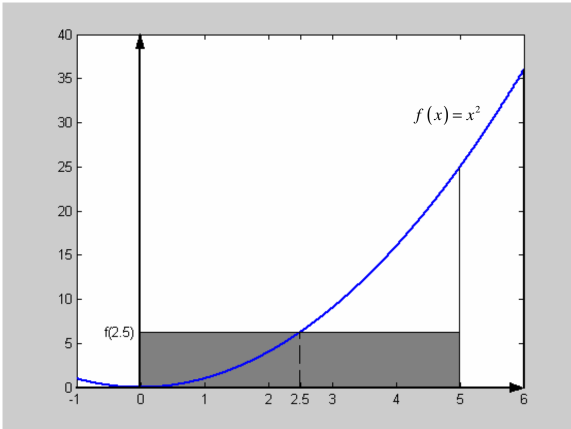


fig. 3.43

So, the shaded area is given by:

$$A=(5-0)\times f(2.5)=31.25.$$

This is not a very accurate approximation (the true answer is 41.67 to 2 d.p.). We can improve the accuracy by splitting the area up into a greater number of rectangles.

Let us split the area under the graph of $f(x)=x^2$ into 5 strips of width one unit, as shown in *fig. 3.44* (these strips are sometimes called **ordinate strips**). We then take the mid point and use the value of the function evaluated at the mid point as the height of each strip. We then add up the area of all the strips to get our approximation.

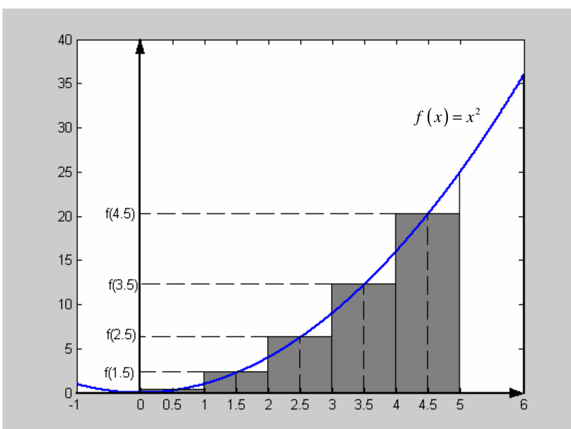


fig. 3.44

Here, the lines $x=0,\ x=1,\ ...,\ x=5$ are the **ordinates** (6 of them). There are 5 ordinate strips.

We can see that the shaded area shown in *fig. 3.44* is given by:

$$A=(1-0)\times f(0.5)+(2-1)\times f(1.5)+(3-2)\times f(2.5)+(4-3)\times f(3.5)+(5-4)\times f(4.5)$$

$$=1\times 0.5^2+1\times 1.5^2+1\times 2.5^2+1\times 3.5^2+1\times 4.5^2$$

$$=41.25.$$

So, with 5 ordinate strips we get a reasonably good result. We could improve the result further by increasing the number of ordinate strips.

Now let us state the general result.

<u>The Mid Point Rule</u>:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{a-b}{n-1} f\left(\frac{x_1+x_2}{2}\right) + \frac{a-b}{n-1} f\left(\frac{x_2+x_3}{2}\right) + \frac{a-b}{n-1} f\left(\frac{x_3+x_4}{2}\right) + \ldots + \frac{a-b}{n-1} f\left(\frac{x_{n-1}+x_n}{2}\right)$$

Where $x_1 = a, x_2, \ldots, x_n = b$ are the ordinates (n of them). There are $n-1$ ordinate strips each of width $\frac{a-b}{n-1}$.

Example 3.37 Use the mid point rule to approximate the area under the graph of $f(x) = x^{\frac{1}{2}} e^x$ from $x = 2$ to $x = 7$ with 6 ordinates (5 ordinate strips).

The 5 ordinates are: $x_1 = 2$, $x_2 = 2 + \left(\frac{7-2}{5}\right) = 3$, $x_3 = 4$, $x_4 = 5$, $x_5 = 6$, $x_6 = 7$. Notice that in general, $x_n = a + nh$, where $h = \frac{a-b}{n-1}$, the width of each ordinate strip.

Using the formula above, we have:

$$\int_2^7 x^{\frac{1}{2}} e^x \mathrm{d}x \approx f\left(\frac{2+3}{2}\right) + f\left(\frac{3+4}{2}\right) + f\left(\frac{4+5}{2}\right) + f\left(\frac{5+6}{2}\right)$$

$$= \sqrt{2.5} \times e^{2.5} + \sqrt{3.5} \times e^{3.5} + \sqrt{4.5} \times e^{4.5} + \sqrt{5.5} \times e^{5.5}$$

$$= 846.02 \text{ to 2 d.p.}$$

Test 3.44 Use the mid point rule to approximate $\int_4^{16} \sqrt{1+x^2}\ \mathrm{d}x$ with 6 ordinate strips (7 ordinates)

Simpson's Rule

Simpson's rule is similar to the mid point rule, except that the function is approximated by a quadratic polynomial between each ordinate strip. We will not give details of the derivation here, we will simply state the result and use it.

Simpson's Rule

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{h}{3}\left[f(x_1)+f(x_n)+4\left(f(x_{\text{even}})\right)+2\left(f(x_{\text{odd}})\right)\right]$$

Where *n* is the number of ordinates, $h=\frac{b-a}{n-1}$ is the width of each ordinate strip, $x_1=a, x_n=b$, $f(x_{\text{even}})$ represents *f* evaluated at each of the even ordinates (except at x_n if *n* is even) and $f(x_{\text{odd}})$ represents *f* evaluated at each of the odd ordinates (except at x_1).

$x_i = x_1 + (i-1)h$.

We can see that $\frac{h}{3}$ multiplies every term in the bracket. Inside the bracket, $f(x_1)$, the function evaluated at the first ordinate, and $f(x_n)$, the function evaluated at the last ordinate, are added together. The function evaluated at each even ordinate (except the last ordinate x_n if *n* is even) is multiplied by 4. The function evaluated at each odd ordinate (except the first ordinate, x_1) is multiplied by 2.

Example 3.38 Using Simpson's rule with 5 ordinates, approximate $\int_0^1 \frac{1}{x^2+1}\ \mathrm{d}x$.

Here we have $f(x)=\frac{1}{x^2+1}$. The lower limit of integration is $a=0$. The upper limit of integration is $b=1$. $h=\frac{1-0}{5-1}=\frac{1}{4}$. So we have $x_1=0$, $x_2=x_1+h=0+\frac{1}{4}=\frac{1}{4}$, $x_3=x_1+2h=0+\frac{2}{4}=\frac{1}{2}$,… and so on. The table below show all the necessary values for the calculation.

x_i	$x_1 = 0$	$x_2 = \frac{1}{4}$	$x_3 = \frac{1}{2}$	$x_4 = \frac{3}{4}$	$x_5 = 1$
$f(x_i)$	$\frac{1}{0^2+1} = 1$	$\frac{1}{\left(\frac{1}{4}\right)^2+1}$	$\frac{1}{\left(\frac{1}{2}\right)^2+1}$	$\frac{1}{\left(\frac{3}{4}\right)^2+1}$	$\frac{1}{2}$

Substituting the above values into Simpson's rule gives:

$$\int_0^1 \frac{1}{x^2+1}\, \mathrm{d}x \approx \frac{1}{12}\left[1 + \frac{4}{\left(\frac{1}{4}\right)^2+1} + \frac{2}{\left(\frac{1}{2}\right)^2+1} + \frac{4}{\left(\frac{3}{4}\right)^2+1} + \frac{1}{2}\right]$$

$$= 0.785 \text{ to 3 d.p.}$$

Example 3.39 Using Simpson's rule with 5 ordinates, approximate $\int_1^3 e^{-2x}\, \mathrm{d}x$.

$f(x_i) = e^{-2x_i}$. $h = \frac{3-1}{5-1} = \frac{1}{2}$.

$x_1 = 1,\ x_2 = x_1 + h = 1 + \frac{1}{2} = 1.5,\ \ x_3 = x_2 + h = 1.5 + \frac{1}{2} = 2,\ \ x_4 = x_3 + h = 2 + \frac{1}{2} = 2.5,\ \ x_5 = x_4 + h = 2.5 + \frac{1}{2} = 3$

x_i	1	1.5	2	2.5	3
$f(x_i)$	e^{-2}	e^{-3}	e^{-4}	e^{-5}	e^{-6}

By Simpson's rule we have

$$\int_1^3 e^{-2x} \approx \frac{1}{3}\times\frac{1}{2}\left(e^{-1} + 4e^{-3} + 2e^{-4} + 4e^{-5} + e^{-6}\right)$$

$$\approx 0.066 \text{ to 3 d.p}$$

Test 3.45 Use Simpson's rule with 4 ordinate strips to find an approximate value for $\int_1^9 \sqrt{\ln x}\, \mathrm{d}x$

Test 3.46 *TRICKY!* You are stranded in a magical forest. The wizard will allow you to return home only if you can tell him an approximate value of $\ln 2$ (a fractional approximation is valid). You cannot remember the value of $\ln 2$. You do not have a calculator – only a stick to write in the soil. How could you come up with an approximation for $\ln 2$? Try it. *Hint*: think of a suitable definite integral and use Simpson's rule. *Hint*: remember, $\ln 1 = 0$. *Answer*: if you use the same method as I did with $n = 7$ you get the fractional approximation $\ln 2 \approx \dfrac{14411}{20790}$.

GCE AS/A Mathematics Chapter 4

4.1 Algebra and Functions

Simplifying Rational Expressions

When working with algebraic rational expressions, it is always worth checking whether any cancellation will lead to a simpler expression. This may include first factorising the numerator and/or denominator where possible.

Example 4.1 Simplify the expression $\frac{6x^2+9x}{4x^2+12x+9}$.

First we notice that both the numerator and the denominator can be factorised,

$$\frac{6x^2+9x}{4x^2+12x+9}=\frac{3x(2x+3)}{(2x+3)(2x+3)}$$

Now we can divide numerator and denominator by $2x+3$,

$$\frac{3x(2x+3)}{(2x+3)(2x+3)}=\frac{3x}{2x+3}.$$

Example 4.2 Simplify the expression $\frac{3x^2-x}{9x^3-6x^2+x}$.

Dividing numerator and denominator by x gives,

$$\frac{3x^2-x}{9x^3-6x^2+x}=\frac{3x-1}{9x^2-6x+1}.$$

Factorising the denominator gives,

$$\frac{3x-1}{9x^2-6x+1}=\frac{3x-1}{(3x-1)^2}=\frac{1}{3x-1}.$$

Test 4.1 Simplify the expression $\frac{3x^2-13x+14}{2x^3-8x}$

Test 4.2 Simplify the expression $\dfrac{9x^2-25}{9x^2+30x+25}$

Algebraic Division

Whenever an algebraic fraction has a numerator with a higher degree (or the same degree) as the denominator, it is possible to divide the denominator into the numerator. For example, $\dfrac{x^5}{x^3}=x^2$. Since the numerator has degree five, which is higher than the degree of the denominator, we can divide the denominator into the numerator. For more complicated examples, there is a method of dividing algebraic quotients which is very similar to the method of long division for numbers. Below is a worked example.

Example 4.3 By dividing the denominator into the numerator, simplify the expression $\dfrac{2x^3-3x^2-2x+2}{x-2}$.

Since the degree of the numerator is three, which is greater than the degree of the denominator, we can divide the denominator into the numerator by dividing 'highest power into highest power' as follows,

$$\begin{array}{r l}
 & 2x^2+x \\
x-2 \big) & 2x^3-3x^2-2x+2 \\
 & \underline{2x^3-4x^2} \\
 & \quad\quad\; x^2-2x+2 \\
 & \quad\quad\; \underline{x^2-2x\quad\;} \\
 & \quad\quad\quad\quad\quad\;\; 2
\end{array}$$

This tells us that $x-2$ goes into $2x^3-3x^2-2x+2$, $2x^2+x$ times with remainder 2, i.e.

$$\frac{2x^3-3x^2-2x+2}{x-2}=2x^2+x+\frac{2}{x-2}.$$

Example 4.4 By dividing the denominator into the numerator, simplify the expression $\dfrac{8x^3-2x^2-3x+3}{2x+1}$.

The working is shown below,

$$
\begin{array}{r}
4x^2 - 3x \\
2x+1 \overline{\big) \; 8x^3 - 2x^2 - 3x + 3} \\
\underline{8x^3 + 4x^2} \qquad\qquad \\
-6x^2 - 3x + 3 \\
\underline{-6x^2 - 3x \quad} \\
3
\end{array}
$$

So, we have that $\frac{8x^3 - 2x^2 - 3x + 3}{2x+1} = 4x^2 - 3x + \frac{3}{2x+1}$.

Partial Fractions

We are easily able to add several fractions together to form one fraction, for example,

$$
\begin{aligned}
\frac{3}{x-2} + \frac{2}{3x+7} &= \frac{3(3x+7) + 2(x-2)}{(x-2)(3x+7)} \\
&= \frac{9x + 21 + 2x - 4}{(x-2)(3x+7)} \\
&= \frac{11x+17}{(x-2)(3x+7)} \\
&= \frac{11x+17}{3x^2 + x - 14}
\end{aligned}
$$

If we were given that fraction $\frac{11x+17}{3x^2+x-14}$ could we split this up into the two fractions we started with? To do this we need to use the method of partial fractions. Partial fractions is essentially the reverse of the process of adding together fractions as shown above. Partial fractions allows us to take an algebraic fraction and split it up into several fractions (where possible). We will look at three basic types of algebraic fractions that can be decomposed by using partial fractions.

<u>TYPE I</u>: The denominator consists of a multiple of linear factors of the form $(ax+b)$.

Example 4.5 The fraction $\frac{7x+3}{(x-1)(2x+3)}$ consists of a multiple of linear factors of the form $(ax+b)$ in the denominator. Notice that the numerator is of a lower degree than the denominator. The fraction can be split up as follows:

$$\frac{7x+3}{(x-1)(2x+3)} \equiv \frac{A}{x-1} + \frac{B}{2x-3}.$$

Our aim is to find *A* and *B*. Adding together the RHS gives:

$$\frac{7x+3}{(x-1)(2x+3)} \equiv \frac{A(2x+3)+B(x-1)}{(x-1)(2x+3)}$$

Now, the above expression is an identity, so we can compare numerators and say that:

$$7x+3 \equiv A(2x+3)+B(x-1)$$

Now we notice that if we set $x=1$, it will eliminate the $(x-1)$ term, and therefore *B*, allowing us to find *A*.

Setting $x=1$ gives:

$$10=5A \quad \Rightarrow \quad A=2$$

So we can now say that

$$7x+3 \equiv 2(2x+3)+B(x-1).$$

Now, we could set $x=-\frac{3}{2}$ to eliminate the $(2x+3)$ term and allow us to find *B*, but there is another technique that we can use. Since we have an identity, for example the number of *x* terms on the LHS must equal the number of *x* terms on the RHS. On the LHS, we have 7 lots of *x*; on the RHS we have $4+B$ lots of *x*. So, we can say:

$7=4+B \quad \Rightarrow \quad B=3$. We call this technique *comparing coefficients*.

We have now solved the problem,

$$\frac{7x+3}{(x-1)(2x+3)} \equiv \frac{2}{x-1} + \frac{3}{2x-3}.$$

Example 4.6 Express $\frac{-11x-19}{2x^2+x-3}$ in partial fraction form.

Our first job is to factorise the denominator.

$$\frac{-11x-19}{2x^2+x-3}=\frac{-11x-19}{(2x+3)(x-1)}.$$

Now we express this in the form

$$\frac{-11x-19}{(2x+3)(x-1)}\equiv\frac{A}{2x+3}+\frac{B}{x-1}$$

Adding the two fractions on the RHS gives,

$$\frac{-11x-19}{(2x+3)(x-1)}\equiv\frac{A(x-1)+B(2x+3)}{(2x+3)(x-1)}$$

Equating numerators gives,

$$-11x-19\equiv A(x-1)+B(2x+3)$$

Setting $x=1$ gives,

$$-30=5B \quad\Rightarrow\quad B=-6$$

Comparing coefficients of x gives,

$$-11=A-12 \quad\Rightarrow\quad A=1$$

So we have,

$$\frac{-11x-19}{(2x+3)(x-1)}\equiv\frac{1}{2x+3}-\frac{6}{x-1}.$$

Test 4.3 Using partial fractions, express $\frac{-7x-26}{2x^2-8}$ in the form $\frac{A}{2x+4}+\frac{B}{x-2}$ where A and B are constants to be determined

Test 4.4 Using partial fractions, express $\frac{1}{x^2-a^2}$, where a is a constant as a sum of two fractions

TYPE II: The denominator consists of a quadratic expression of the form ax^2+bx+c which cannot be factorised into linear factors.

Example 4.7 Split $\dfrac{9x^2-35x+8}{(x-4)(x^2-2x-2)}$ up into partial fractions.

Notice that the denominator contains the term x^2-2x-2 which cannot be factorised. This type of fraction can be split up into the following form:

$$\frac{9x^2-35x+8}{(x-4)(x^2-2x-2)} \equiv \frac{A}{x-4}+\frac{Bx+C}{x^2-2x-2}$$

$$\equiv \frac{A(x^2-2x-2)+(Bx+C)(x-4)}{(x-4)(x^2-2x-2)}$$

So we must have that $9x^2-35x+8 \equiv A(x^2-2x-2)+(Bx+C)(x-4)$.

Setting $x=4$ gives,

$12=6A \quad \Rightarrow \quad A=2$.

Comparing coefficients of x^2 gives,

$9=A+B \quad \Rightarrow \quad B=7$.

Comparing constants gives,

$8=-2A-4C \quad \Rightarrow \quad C=-3$.

S, we have found that

$$\frac{9x^2-35x+8}{(x-4)(x^2-2x-2)} \equiv \frac{2}{x-4}+\frac{7x-3}{x^2-2x-2}.$$

Test 4.4 Write the fraction $\dfrac{3x^2-8x+7}{(2x-1)(3x^2-x+5)}$ in the form $\dfrac{A}{2x-1}+\dfrac{Bx+C}{3x^2-x+5}$ where A and B are constants to be determined

<u>TYPE III</u>: The denominator contains repeated linear terms, for example $(ax+b)^2$.

Example 4.8 Express $\dfrac{2x^2+29x-11}{(2x+1)(x-2)^2}$ in partial fractions.

In this example, the partial fraction form is as follows,

$$\frac{2x^2+29x-11}{(2x+1)(x-2)^2} \equiv \frac{A}{2x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

In the denominators on the RHS, we have a linear factor on its own, $x-2$, and a repeated linear factor, $(x-2)^2$. Adding together the RHS gives,

$$\frac{2x^2+29x-11}{(2x+1)(x-2)^2} \equiv \frac{A(x-2)^2+B(2x+1)(x-2)+C(2x+1)}{(2x+1)(x-2)^2}$$

So we must have,

$$2x^2+29x-11 \equiv A(x-2)^2+B(2x+1)(x-2)+C(2x+1).$$

Setting $x=-\frac{1}{2}$ gives,

$$2\left(-\frac{1}{2}\right)^2-\frac{29}{2}-11=A\left(-\frac{5}{2}\right)^2 \quad \Rightarrow \quad A=-4 \text{ (after some work)}$$

Setting $x=2$ gives us that $C=11$ (after some work).

Comparing coefficients of x^2 gives,

$$2=A+2B \quad \Rightarrow \quad B=3.$$

So we have found that,

$$\frac{2x^2+29x-11}{(2x+1)(x-2)^2} \equiv -\frac{4}{2x+1}+\frac{3}{x-2}+\frac{11}{(x-2)^2}.$$

Test 4.5 Express the fraction $\frac{18x+20}{(3x+4)^2}$ in the form $\frac{A}{3x+4}+\frac{Bx+C}{(3x+4)^2}$, where A, B and C are constants to be determined

Test 4.6 Express the fraction $\frac{8x^2+x-3}{(x+2)(x-1)^2}$ in the form $\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$, where A, B and C are constants to be determined

Test 4.7 Given that $\frac{5x^2-13x+5}{(x-2)^3}\equiv\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$, where A, B and C are constants, find the values of A, B and C

An important point to make is that in order to express a quotient in partial fraction form, the numerator must be at least one degree less than the denominator. If this is not the case we must first ‘do the division’ as illustrated in the next example.

Example 4.9 Express $\frac{x^2+x-5}{x^2-2x-3}$ in partial fraction form.

Since the numerator is not of a lower order than the denominator, we can divide the denominator into the numerator.

$$\begin{array}{r} 1 \\ x^2-2x-3\overline{)x^2+x-5} \\ \underline{x^2-2x-3} \\ 3x-2 \end{array}$$

i.e. $\frac{x^2+x-5}{x^2-2x-3}\equiv 1+\frac{3x-2}{x^2-2x-3}$.

Now we can work on the fraction $\frac{3x-2}{x^2-2x-3}$ by factorising the denominator and using partial fractions in the usual way.

Below is a summary of the partial fraction forms.

Denominator containing	Expression	Form of partial fractions
Linear factors	$\frac{f(x)}{(x+a)(x+b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x+b)}+\frac{C}{(x+c)}$
Repeated linear factors	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
Quadratic factors	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$
General example	$\frac{f(x)}{(x^2+a)(x+b)^2(x+c)}$	$\frac{Ax+b}{(x^2+a)}+\frac{C}{(x+b)}+\frac{D}{(x+b)^2}+\frac{E}{(x+c)}$

4.2 Coordinate Geometry in the (x,y) Plane

Parametric and Cartesian Equations

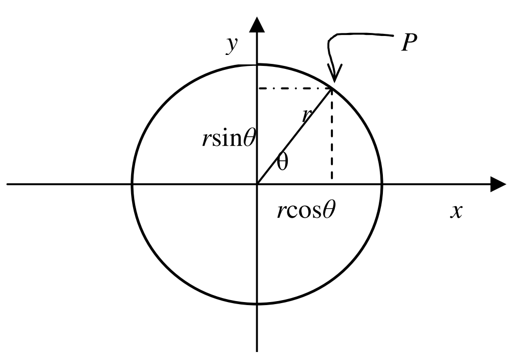


fig. 4.1

Consider a circle with radius r and centre at the origin. We have seen that we can express any point on the circle, for example the point P, in terms of it's x-y coordinates. We can define the position of P by writing the equation of the circle in the form $x^2 + y^2 = r$. This is called a *Cartesian* equation (after the French philosopher and mathematician René Descartes, 1596 – 1650). It is an equation which defines the relationship between the x and y coordinates (and some constant, r).

However, there is an alternative method of defining the position of P. We can define the x and y coordinates of P by defining the distance in the x and y direction from the origin in terms of the radius r and a new variable, θ, which is the angle the radius makes with the x-axis. We can see that:

$x = r\cos\theta$ and $y = r\sin\theta$.

This new way of expressing the position of a point on a circle involves a new variable, θ. θ is called a *parameter*, since it is a variable which appears both in the expression for the x-coordinate and the y-coordinate. The previous equation is an example of a *parametric equation* (with parameter θ). Sometimes it turns out to be more convenient to work with parametric equations rather that Cartesian equations, and sometimes it is necessary to convert from one form to another, depending on which form is most convenient for a particular calculation. We introduced parametric equations with the example of a circle; it is also possible to paramatise parabolas and straight lines as shown in the next examples.

Example 4.1 Find the Cartesian form of the following pair of parametric equations,

$$x = t - 2 \qquad y = 2t - 9.$$

In questions like this, we usually proceed by using one of the equations to express t in terms of either x or y and then substitute this for t in the other equation. In this example, let us use the first equation to write t in terms of x. We can see that,

$$t = x + 2.$$

Substituting this for t in the second equation gives,

$$y = 2(x+2) - 9$$

$$= 2x + 4 - 9$$

$$= 2x - 5$$

So, we have arrived at the Cartesian form of the equation, $y = 2x - 5$.

Example 4.2 Find the Cartesian form of the following pair of parametric equations,

$$x = 2t - 1 \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$
$$y = 12t^2 - 14t + 6 \ldots\ldots\ldots..(2)$$

In this example, it is easiest to express t in terms of x using (1). Doing this we get,

$t = \dfrac{x+1}{2}$. Substituting for t in (2) gives,

$$y = 12\left(\frac{x+1}{2}\right)^2 - 14\left(\frac{x+1}{2}\right) + 6$$

$$=12\left(\frac{x^2+2x+1}{4}\right)-7x-7+6$$

$$=3x^2+6x+3-7x-7+6$$

$$=3x^2-x+2$$

So, we have arrived at the Cartesian form, $y=3x^2-x+2$.

Example 4.3 Find the Cartesian form of the following pair of parametric equations,

$$x=\frac{1}{1-t}\ \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$y=\frac{1}{7t-5}\ \ldots\ldots\ldots\ldots\ldots..(2)$$

From (1) we can see that $x(1-t)=1 \quad \Rightarrow \quad x-tx=1 \quad \Rightarrow \quad t=\frac{x-1}{x}$. Substituting this into (2) gives,

$$y=\frac{1}{7\left(\frac{x-1}{x}\right)-5}=\frac{1}{\frac{7x-7}{x}-5}$$

$$=\frac{1}{\frac{7x-7-5x}{x}}=\frac{x}{2x-7}$$. So, we have arrived at the Cartesian form, $y=\frac{x}{2x-7}$.

Test 4.1 Find the Cartesian form of the following pair of parametric equations, $x=2\sqrt{t}$ and $y=8t^2+5$

Test 4.2 Find the Cartesian form of the following pair of parametric equations, $x=\frac{t}{1-3t}$ and $y=\frac{t}{1+2t}$

4.3 Sequences and Series

Binomial Expansion

Introduction

Expand the following: $(1+x)^2$, $(1+x)^3$, $(1+x)^4$. We can see that the expansion of $(1+x)^n$ becomes very laborious as n increases beyond 3. Clearly, trying to expand $(1+x)^7$ would be a very long and boring calculation. By the end of this section, we will be able to calculate $(1+x)^7$ quickly and easily.

In the three expansion questions posed above, you should see that the constant in each expansion is 1. Why is this? Let us look at the example $(1+x)^2$,

$$(1+x)^2 = (1+x)(1+x) = 1+x+x+x^2$$

There is only one way in which we can get a constant term, and that is from the two constant terms from the liner factors multiplied together, as shown above.

$$(1+x)^2 = (1+x)(1+x) = 1+x+x+x^2$$

There are two ways in which we can get an x term, as shown above.

$$(1+x)^2 = (1+x)(1+x) = 1+x+x+x^2$$

There is one way in which we can get an x^2 term, as shown above.

In calculating $(1+x)^2$, there are a total of four multiplications needed, as indicated by the four arrows in the diagrams above. Thus we have found that there is one way (out of four total multiplications) in which we can get a constant, two ways (out of four total multiplications) in which we can get an x terms and one way (out of four total multiplications) in which we can get an x^2 term. You may like to consider the case $(1+x)^3$ and see how many multiplications are necessary to evaluate this. You can also work out how many different ways you can get constant terms, x terms, x^2 and x^3 terms. It turns out that there is a special pattern between the coefficients of the powers of x in the expansion of $(1+x)^n$. We will look at this pattern next.

Pascal's Triangle

fig. 4.2 shows the first part of Pascal's triangle (named after the French mathematician and philosopher Blaise Pascal (1623 – 1662). Each number in the triangle is obtained by adding together the two numbers directly above.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

fig. 4.2

The second row is 1 1; these are the coefficients of $(1+x)^1$. The third row is 1 2 1, these are the coefficients of $(1+x)^2 = 1+2x+1$. The fourth row is 1 3 3 1, these are the coefficients of $(1+x)^3 = 1+3x+3x^2+x^3$. From Pascal's triangle, what is the expansion of $(1+x)^6$? From the triangle, we can see that the coefficients are 1 6 15 20 15 6 1, so the expansion is $(1+x)^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6$. By filling in the next row of Pascal's triangle, write down the expansion of $(1+x)^7$. Pascal's triangle can also be used to expand an expression of the form $(a \pm bx)^n$, for example $(2-3x)^5$ may be written as $2^5\left(1+\left(-\frac{3x}{5}\right)\right)^5$, we can write X in place of $\left(-\frac{3x}{5}\right)$, so we have $(2-3x)^5 = 2^5(1+X)^5$ and expand this in the usual way, remembering to replace X with $\left(-\frac{3x}{5}\right)$ at the end. We will now develop a formula for expanding $(1+x)^n$, but first we need some to introduce some new concepts.

Binomial Coefficients

The *factorial* of a positive integer, n, denoted $n!$ is defined by,

$$n! = n(n-1)(n-2)(n-3)\ldots 1$$

For example, $3! = 3\times2\times1 = 6$, $4! = 4\times3\times2\times1 = 24$. We define the factorial of zero to be one, i.e. $0! = 1$.

Next we introduce the *binomial coefficient* notation,

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}$$

For example,

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 10$$

$$\binom{6}{1} = \frac{6!}{1!(6-1)!} = \frac{6\times5\times4\times3\times2\times1}{1\times5\times4\times3\times2\times1} = 6$$

The binomial coefficients $\binom{a}{b}$ give the number of ways of choosing *a* objects from a set of *b* objects. For example, in how many ways can a committee of 4 people be formed from a group of 9 people? The answer is $\binom{9}{4} = 126$ ways.

The binomial coefficient $\binom{n}{r}$ is sometimes written as ${}^nC_r$. On your calculator you should find a button labeled [${}^nC_r$] or [nCr]. For example, pressing [9] followed by [${}^nC_r$] followed by [4] on your calculator should return the answer 126.

Test 4.2 Calculate $\binom{4}{2}$ and $\binom{7}{3}$

Binomial Expansion

We are now ready to state the general formula. We have not proved this formula here, but you can see from the previous examples that a detailed combinatorial analysis of the expansion of $(1+x)^n$ is possible. Such a detailed analysis leads to the following result:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$$

This can be written in sigma notation as,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \quad \ldots\ldots\ldots\ldots\ldots\ldots(\dagger)$$

This can also be equivalently written as,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots + x^n$$

Check that you can get from (†) to the above form.

We have already commented that any expression of the form $(a \pm bx)^n$ can be expressed in the form $A(1 \pm Bx)^n$, and so the above formulas can be used to expand anything in the form $(a \pm bx)^n$. Sometimes, however, the result for the expansion of $(a+bx)^n$ is stated separately (*b* may be negative). This result is stated below,

$$(a+bx)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} (bx)^r$$

Or, alternatively,

$$(a+bx)^n = a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}(bx)^4 + \ldots + (bx)^n$$

Example 4.4 Expand $(2+x)^6$.

There is more than one way to do this. You could use Pascal's triangle for example, but we will use the general formula we have stated above.

$$(2+x)^6 = 2^6 + 6\times 2^5 \times x + \frac{6\times 5}{2!}\times 2^4 \times x^2 + \frac{6\times 5\times 4}{3!}\times 2^3 \times x^3 + \frac{6\times 5\times 4\times 3}{4!}\times 2^2 \times x^4 + \frac{6\times 5\times 4\times 3\times 2}{5!}\times 2\times x^5 + \frac{6\times 5\times 4\times 3\times 2\times 1}{6!}\times x^6$$

$$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$$

Example 4.5 Expand $(1-2x)^5$.

It is easy to make a mistake when the *x* term inside the bracket is negative. Make sure you use brackets correctly and the whole term, $-2x$, gets raised to a power in each step of the calculation.

$$(1-2x)^5 = 1+(-2x)+\frac{5\times4}{2!}\times(-2x)^2+\frac{5\times4\times3}{3!}\times(-2x)^3+\frac{5\times4\times3\times2}{4!}\times(-2x)^4+(-2x)^5$$

$$=1-2x+40x^2-80x^3+80x^4-32x^5$$

Notice that when the *x* term inside the bracket is negative, we get an alternating series as the answer (since a negative number raised to an even power is positive, whilst a negative number raised to an odd power is negative). It is worth while checking this when you have finished a problem of this type, to make sure you have not made a mistake with the signs.

Example 4.6 What is the coefficient of x^6 in the expansion of $(3-x)^9$?

It would be a waste of time to expand the whole thing in this example, since we are only interested in the coefficient of x^6, and none of the other terms. If you study the general formula we stated earlier, it should be apparent to you that the term involving x^6 is $\frac{9\times8\times7\times6\times5\times4}{6!}\times3^{9-6}\times(-x)^6$ and so the coefficient of x^6 is,

$$\frac{9\times8\times7\times6\times5\times4}{6!}\times3^3 = 2268.$$

Test 4.3 Expand $\left(3-\frac{x}{2}\right)^6$ fully

Test 4.4 What is the coefficient of x^5 in the expansion of $(x-1)^6$?

Let us recall our general binomial expansion formula,

$$(a+bx)^n = a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}(bx)^4 + \ldots + (bx)^n$$

So far, we have only considered this expansion for *n* a positive integer. What if *n* is not a positive integer? Well, it turns out that the above formula is valid for all rational *n* (positive or negative), provided $|x| < \frac{a}{b}$ (this means that $-\frac{a}{b} < x < \frac{a}{b}$). The extra condition $|x| < a$ is necessary for the series to converge. Otherwise, the formula works in exactly the same way.

Example 4.7 By using the binomial expansion, find a polynomial approximation for $(1+x)^{-1}$.

Using the formula above, we have,

$$(1+x)^{-1} = 1+(-1)\times 1^{-1-1}\times x+\frac{-1\times -2}{2!}\times 1^{-1-2}\times x^2+\frac{-1\times -2\times -3}{3!}\times 1^{-1-3}\times x^3+\frac{-1\times -2\times -3\times -4}{4!}\times 1^{-1-4}\times x^4+\frac{-1\times -2\times -3\times -4\times -5}{5!}\times 1^{-1-4}\times x^5+\ldots$$

$$=1-x+x^2-x^3+x^4-x^5+\ldots$$

Notice that when n was a positive integer, the binomial expansion formula came to a natural end, but when n is not a positive integer, the expansion has no end, it is an infinite expansion. Remember to state in your answer that this is only valid for $|x|<1$.

Example 4.8 By using the binomial expansion, find the first four terms in the polynomial approximation for $\frac{1}{(2-3x)^2}$.

First of all we write $\frac{1}{(2-3x)^2}$ as $(2-3x)^{-2}$. Now we use the formula,

$$(2-3x)^{-2} \approx 2^{-2}+(-2)\times 2^{-2-1}\times(-3)\times x+\frac{-2\times -3}{2!}\times 2^{-2-2}\times(-3\times x)^2+\frac{-2\times -3\times -4}{3!}\times 2^{-2-3}\times(-3\times x)^3$$

$$=\frac{1}{4}+2x+\frac{27}{16}x^2+\frac{27}{4}x^3$$

This is valid for $|3x|<2$, i.e. for $|x|<\frac{2}{3}$.

Example 4.9 By using the binomial expansion, find the first four terms in the polynomial approximation for $\sqrt{1-x}$.

First we write $\sqrt{1-x}$ as $(1-x)^{\frac{1}{2}}$. Then we use the formula,

$$(1-x)^{\frac{1}{2}} \approx 1^{\frac{1}{2}}+\frac{1}{2}\times 1^{\frac{1}{2}-1}\times(-1\times x)+\frac{\frac{1}{2}\times\left(\frac{1}{2}-1\right)}{2!}\times 1^{\frac{1}{2}-2}\times(-1\times x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\times 1^{\frac{1}{2}-3}\times(-1\times x)^3$$

$$=-\frac{1}{2}x-\frac{1}{16}x^2-\frac{1}{16}x^3$$

This is valid for $|x|<1$

Test 4.5 By using the binomial expansion, find the first four terms in the polynomial approximation for $\dfrac{2}{(2-x)^3}$

Test 4.6 By using the binomial expansion, find the first four terms in the polynomial approximation for $\dfrac{1}{2\sqrt{1+\frac{x}{2}}}$

Series Expansion of Rational Functions

We are now able to find a series expansion for $(a+bx)^n$ for any rational number *n*. Combining this with our knowledge of partial fractions, we are able to find series expansions of some rational functions, as illustrated below.

Example 4.10 Find a series expansion for the rational expression $f(x)=\dfrac{3x+5}{x^2+2x-3}$.

Using the methods developed in the section on partial fractions, it is possible to write $f(x)$ in the following way, $f(x)=\dfrac{1}{x+3}+\dfrac{2}{x-1}$ (check!). We can find series expansions of $\dfrac{1}{x+3}=(x+3)^{-1}$ and $\dfrac{2}{x-1}=2(x-1)^{-1}$, so adding these two series expansions together will give us a series expansion for $f(x)$. We have that,

$$(x+3)^{-1}=\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2-\frac{1}{81}x^3+\frac{1}{243}x^4+\ldots \qquad \text{(check!)}$$

This is valid for $|x|<3$

and,

$$2(x-1)^{-1}=2(1-x+x^2-x^3+x^4-\ldots)=2-2x+2x^2-2x^3+2x^4-\ldots \qquad \text{(check!)}$$

This is valid for $|x|<1$

So,

$$f(x)=(x+3)^{-1}+2(x-1)^{-1}=\left(\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2-\frac{1}{81}x^3+\frac{1}{243}x^4+\ldots\right)+\left(2-2x+2x^2-2x^3+2x^4-\ldots\right)$$

$$=\frac{7}{3}-\frac{19}{9}x+\frac{55}{27}x^2-\frac{163}{81}x^3+\frac{587}{243}x^4+\ldots$$

Now, we must state for which values this expansion is valid. The expansion of $(x+3)^{-1}$ is valid for $|x|<3$; the expansion of $2(x-1)^{-1}$ is valid for $|x|<1$, so both expansions are valid for $|x|<1$, hence the final answer above is valid for $|x|<1$.

Test 4.7 Express $\dfrac{1-x-x^2}{(1-2x)(1-x)^2}$ as the sum of three partial fractions. Hence, expand this expression in ascending powers of x up to and including the term in x^3. State the range of values of x for which the full expansion is valid.

4.4 Trigonometry

sin, cos and tan of pi/3, pi/4, pi/6

When we calculate the sine, cosine or tangent of an angle on our calculator, it is not usually possible to write down the exact answer, since it will usually be an infinite decimal expansion. There are a few special angles, however that have a sine, cosine, and tangent that can be expressed exactly, for example $\tan\left(\frac{\pi}{4}\right)=1$

Consider a right angled triangle *ABC* with $\angle CAB=\frac{\pi}{2}$. Let us set both of the other two angles, $\angle ABC$ and $\angle BCA$ to $\frac{\pi}{4}$ (so that at the three interior angles add up to π). Now since $\angle ABC=\angle CAB$, we must have that the side *AB* is equal in length to the side *AC*, let us set this to 1. From Pythagoras, we can say that the length of the hypotenuse, *BC* is equal to $\sqrt{2}$. This information is shown in *fig. 4.3*.

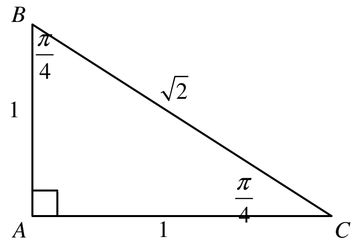


fig. 4.3

From this diagram, we can write down *exact* values for the sine, cosine and tangent of $\frac{\pi}{4}$. The results are as follows,

$$\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \qquad \tan\left(\frac{\pi}{4}\right)=1$$

Consider the right angled triangle *DEF* with $\angle FDE=\frac{\pi}{2}$, $\angle EFD=\frac{\pi}{3}$, $\angle DEF=\frac{\pi}{6}$ (notice that the interior angles to add up to π). Let us set the length of the hypotenuse, *EF* equal to 2 and the length of the side *DF* equal to 1. From Pythagoras then, the length of the other side, *DE* must be equal to $\sqrt{3}$. This information is shown in *fig. 4.4*.

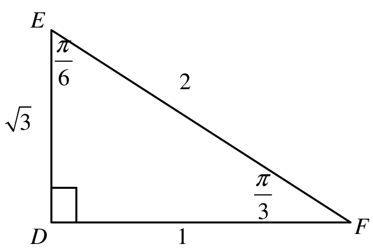


fig. 4.4

From this diagram, we can write down *exact* values for the sine, cosine and tangent of $\frac{\pi}{3}$ and $\frac{\pi}{6}$. The results are as follows,

$$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right)=\frac{1}{2} \qquad \tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2} \qquad \cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \qquad \tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$$

More Trigonometric Identities

In this section, we will learn some more trig identities. The first of these are stated below, but we will not prove it here. It is not difficult to prove (at least for acute angles) and such a proof can be found in A-level text books.

Compound Angles

$$\sin(A\pm B)\equiv\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)\equiv\cos A\cos B\mp\sin A\sin B$$

Example 4.11 Without using a calculator, write down the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

It seems that *fig. 4.3* and *fig. 4.4* are not much use to us here, since we are interested in an angle of $\frac{5\pi}{12}$. These diagrams do, however, allow us to write down exact values of the sine of $\frac{\pi}{4}$ and $\frac{\pi}{6}$, and we notice that $\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$. So, we can use *fig. 4.3* and *fig. 4.4* along with the formula for $\sin(A \pm B)$ above to solve this problem,

$$\sin\left(\frac{5\pi}{12}\right)=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}=\frac{1+\sqrt{3}}{2\sqrt{2}}$$

rationalising

the denominator $\qquad =\frac{\sqrt{2}+\sqrt{6}}{4}$

Test 4.7 Without using a calculator, write down the exact value of $\cos\left(\frac{\pi}{6}\right)$

Double Angle Formula

We have stated the identity $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$. From this, we can write down an identity for $\sin(2\theta)$ as follows,

$$\sin(2\theta)=\sin(\theta+\theta)\equiv\sin\theta\cos\theta\pm\cos\theta\sin\theta$$

$$=2\sin\theta\cos\theta\,.$$

So, we have the identity,

$$\sin(2\theta)\equiv 2\sin\theta\cos\theta$$

Similarly, by using the identity $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$, we can derive the following cosine double angle formula,

$$\cos(2\theta) \equiv \cos^2\theta - \sin^2\theta$$

Since $\sin^2\theta \equiv 1-\cos^2\theta$, we can write the above identity as,

$$\cos(2\theta) = \cos^2\theta - \left(1-\cos^2\theta\right)$$

so,

$$\cos(2\theta) \equiv 2\cos^2\theta - 1$$

Since $\cos^2\theta \equiv 1-\sin^2\theta$, we can write the above as,

$$\cos(2\theta) = 2\left(1-\sin^2\theta\right) - 1$$

so,

$$\cos(2\theta) \equiv 1-\sin^2\theta$$

So, we now have three different expressions for $\cos(2\theta)$.

Test 4.9 By using the identity $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$, write down an identity for $\sin(3\theta)$ in its simplest form (there is more than one acceptable answer to this question)

We can also derive a double angle formula for tangent, by remembering that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$. We have,

$$\tan(2\theta) \equiv \frac{\sin(2\theta)}{\cos(2\theta)} \equiv \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}.$$

Dividing numerator and denominator by $\cos^2\theta$ gives,

$$\tan(2\theta) \equiv \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}} \equiv \frac{2\tan\theta}{1-\tan^2\theta}.$$ So, we have the identity,

$$\tan(2\theta) \equiv \frac{2\tan\theta}{1-\tan^2\theta}$$

Factor Formulae

We have the following four identities,

$$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These identities can be proved by using the previously stated identities. We will prove the first one here, and leave the rest as an exercise. They are all proved in a similar way.

We have that

$$\sin(\theta+\varphi) \equiv \sin\theta\cos\varphi + \cos\theta\sin\varphi$$

and

$$\sin(\theta-\varphi) \equiv \sin\theta\cos\varphi - \cos\theta\sin\varphi$$

Adding these two equations gives,

$$\sin(\theta+\varphi)+\sin(\theta-\varphi) \equiv 2\sin\theta\cos\varphi \ldots\ldots.(\blacklozenge)$$

If we now let $\theta+\varphi = A$ and $\theta-\varphi = B$

We can see that $\theta = \dfrac{A+B}{2}$ and $\varphi = \dfrac{A-B}{2}$. Substituting this into (♦) gives,

$$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \text{ as required.}$$

Proving Trigonometric Identities

We now have a collection of standard trigonometric identities which we can use to solve problems and prove further identities. The identities that we have stated so far by no means make up a list of all the trigonometric identities that exist. The identities we have mentioned so far do enable us, however, to prove many more results. Below is an example of this in action.

Example 4.12 Prove the identity $\sqrt{\dfrac{1+\tan^2\theta}{1+\cot^2\theta}} \equiv \tan\theta$.

As usual, we will work on the more complicated side and make it look like the simpler side. We have the identity $1+\tan^2\theta = \sec^2\theta$ and $1+\cot^2\theta = \text{cosec}^2\theta$, from section 3.2. Substituting these into the LHS gives,

$$\text{LHS} = \sqrt{\frac{\sec^2\theta}{\text{cosec}^2\theta}} = \frac{\sec\theta}{\text{cosec}\,\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{RHS}\text{, as required}$$

Example 4.13 Prove the identity $\sin 3\theta + \sin\theta \equiv 4\sin\theta\cos^2\theta$.

In this example, we will work on the LHS and show that it is the same as the RHS. We use the identity $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ to write down an identity for $\sin 3\theta$ (this is test 4.9). We have,

$$\sin(2\theta+\theta) \equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta.$$

Next, we use the identity $\sin(2\theta) \equiv 2\sin\theta\cos\theta$ top replace the $\sin 2\theta$ term and the identity $\cos(2\theta) \equiv 2\cos^2\theta - 1$ to replace the $\cos 2\theta$ term. (Note that there is more than one identity for $\cos 2\theta$ to choose from, but since we are trying to make this look something like $\sin\theta\cos^2\theta$, this is the only one that will work in this case). Doing this gives,

$$\sin 3\theta \equiv 2\sin\theta\cos\theta\cos\theta + \left(2\cos^2\theta - 1\right)\sin\theta$$

$$\Rightarrow \quad \sin 3\theta \equiv 2\sin\theta\cos^2\theta + 2\cos^2\theta\sin\theta - \sin\theta$$

$$\Rightarrow \quad \sin 3\theta \equiv 4\sin\theta\cos^2\theta - \sin\theta$$

So, $\sin 3\theta + \sin\theta \equiv 4\sin\theta\cos^2\theta$ as required.

Test 4.10 Prove the identity $\sin(A+B) - \sin(A-B) \equiv 2\cos A\sin B$

Test 4.11 Prove the identity $\dfrac{\sin A + \sin B}{\cos A + \cos B} \equiv \tan\left(\dfrac{A+B}{2}\right)$

Solving Trigonometric Equations

Trigonometric identities are also useful when solving trigonometric equations. Here are some examples.

Example 4.14 Solve the equation $\cos 4\theta - \cos\theta = 0$ in the interval $0 \le \theta \le \pi$.
When solving equations of this type, we need to use the factor formula to express the LHS as a product of two trigonometric functions. In this example, we use the identity $\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ to write the LHS as,

$\cos 4\theta - \cos\theta = -2\sin\left(\frac{5\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$. So now we need to solve the equation,

$$-2\sin\left(\frac{5\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) = 0 \quad \Rightarrow \quad \sin\left(\frac{5\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) = 0.$$

We can solve the equation in this form, since the LHS is zero if and only if at least one of the terms $\sin\left(\frac{5\theta}{2}\right)$ or $\sin\left(\frac{3\theta}{2}\right)$ are zero.

Setting $\sin\left(\frac{5\theta}{2}\right) = 0$ gives the solutions $\frac{5\theta}{2} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots \quad \Rightarrow \theta = 0,\ \frac{2\pi}{5},\ \frac{4\pi}{5}$ in the specified range.

Setting $\sin\left(\frac{3\theta}{2}\right)$ gives the solutions $\frac{3\theta}{2} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots \quad \Rightarrow \theta = 0,\ \frac{2\pi}{3}$ in the specified range.

So, all together we have the solutions, $\quad \Rightarrow \theta = 0,\ \frac{2\pi}{5},\ \frac{2\pi}{3},\ \frac{4\pi}{5}$ for $0 \le \theta \le \pi$.

> **Test 4.12** Solve the equation $\sin 7\theta = \sin 3\theta$ in the interval $0 \le \theta \le \pi$

Example 4.15 Solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$ for $0 \le \theta \le 2\pi$.

Here, the arguments of the trig functions are different. Our aim is to express this equation as an equation in $\cos\theta$ by using the identity $\cos(2\theta) \equiv 2\cos^2\theta - 1$, as follows.

$$4\cos 2\theta - 2\cos\theta + 3 = 0$$

$\Rightarrow\ 4\left(2\cos^2\theta-1\right)-2\cos\theta+3=0$

$\Rightarrow\ 8\cos^2\theta-4-2\cos\theta+3=0$

$\Rightarrow\ 8\cos^2\theta-2\cos\theta-1=0$. This is now a quadratic equation in $\cos\theta$ and it factorises as follows,

$8\cos^2\theta-2\cos\theta-1=0 \quad\Leftrightarrow\quad (4\cos\theta+1)(2\cos\theta-1)=0$

Setting $4\cos\theta+1=0 \ \Rightarrow\ \cos\theta=-\frac{1}{4}$, which has solutions $\theta=1.823$ and $\theta=4.460$ for $0\le\theta\le 2\pi$.

Setting $2\cos\theta-1=0 \ \Rightarrow\ \cos\theta=\frac{1}{2}$, which has solutions $\theta=\frac{\pi}{3}$ and $\theta=\frac{5\pi}{3}$ for $0\le\theta\le 2\pi$.

So, all together we have the solutions $\theta=\frac{\pi}{3}$, 1.823, 4.460, $\frac{5\pi}{3}$ for $0\le\theta\le 2\pi$.

Test 4.13 Solve the equation $4\cos\theta=3\sin 2\theta$ for $0\le\theta\le 2\pi$

Equations of the Form *a*cos*x*+*b*sin*x*=*c*

Example 4.16 Solve the equation $3\cos x+4\sin x=5$ for $0\le\theta\le 360^\circ$.

For equations of this type, we write the LHS in the form $R\sin(x+\phi)$ or $R\cos(x+\phi)$ (either form will work) where R is a constant greater than zero and ϕ is an acute angle. Let us solve this example by expressing $3\cos x+4\sin x$ in the form $R\sin(x+\phi)$. We write,

$$3\cos x+4\sin x\equiv R\sin(x+\phi)$$

$$\equiv R\sin x\cos\phi+R\cos x\sin\phi$$

Comparing coefficients of $\cos x$ yields, $3=R\sin\phi$ ....(**1**)

Comparing coefficients of $\sin x$ yields, $4=R\cos\phi$ ....(**2**)

Squaring both (**1**) and (**2**) and adding gives,

$R^2 \sin^2\phi + R^2\cos^2\phi = 3^2 + 4^2$

$\Rightarrow \quad R^2\left(\sin^2\phi + \cos^2\phi\right) = 25 \quad \Rightarrow \quad R = 5$

Dividing (**2**) by (**1**) gives,

$\dfrac{R\sin\phi}{R\cos\phi} = \dfrac{4}{5} \quad \Rightarrow \quad \tan\phi = \dfrac{4}{5} \quad \Rightarrow \quad \phi = 53.130^\circ$ (we take ϕ to be the acute angle with a tan of 4/5)

So we have found that,

$3\cos x + 4\sin x = 5 \quad \Leftrightarrow \quad 5\sin\left(x + 53.130^\circ\right) = 5 \quad \Rightarrow \quad \sin\left(x + 53.130^\circ\right) = 1$

We can now solve this to give

$x + 53.130^\circ = \sin^{-1} 1 \quad \Rightarrow \quad x + 53.130^\circ = 90,\ 180,\ 270,\ 360,...$

$\Rightarrow \quad x = 36.9^\circ,\ 126.9^\circ,\ 216.9^\circ,\ 306.9^\circ$ for $0 \le \theta \le 360^\circ$.

This is a standard method. Once you have seen one example, you can solve other examples by following the method through. We just have to write down the identity $a\cos x + b\sin x \equiv R\sin\left(x+\phi\right)$, expand the RHS, compare coefficients to give two equations, square and add these two equations to find *R*, divide the two equations to find ϕ (remember we always take the acute angle for ϕ), then we can solve the equation $R\sin\left(x+\phi\right)$.

Test 4.14 Solve the equation $5\sin\theta + 12\cos\theta = 7$ for $0 \le \theta \le 360^\circ$

4.5 Differentiation and Integration

(First Order, Separable) Differential Equations

Differential equations are equations which involve a derivative of a variable. For example, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x$ is a differential equation. Can you write down an expression for *y* for this simple example? It is not difficult to see that we could have $y = \dfrac{1}{2}x^2$. We say that $y = \dfrac{1}{2}x^2$ is a *solution* of the differential equation $\frac{dy}{dx} = x$. We can see that $y = \frac{1}{2}x^2 - 12$ is also a solution of the differential equation $\frac{dy}{dx} = x$. In fact, $y = \frac{1}{2}x^2 + c$ is a solution of the differential equation $\frac{dy}{dx} = x$ for any constant, c. So, we can see that differential equations do not have unique solutions (unless we are given some extra information). For the differential equation above, it was easy to see that $y = \frac{1}{2}x^2$ is a solution (how did you arrive at this answer?). For more complicated cases, we will need to develop a more systematic method for solving differential equations. Solving differential equations, in general, is a complicated business and there are many methods for solving different types of differential equations. Here we will barely scratch the surface and consider very simple, so called, separable, first order differential equations.

A differential equation which can be written in the form $\frac{dy}{dx} = f(x)$ for some function $f(x)$ is called a separable (first order) differential equation, and we can solve it by direct integration (provided we know how to integrate $f(x)$).

Example 4.17 Solve the differential equation $\frac{dy}{dx} = 3x^2 + 2$.

Integrating both sides with respect to x gives, $\int \frac{dy}{dx} dx = \int 3x^2 + 2dx$

$\Rightarrow \quad y = x^3 + 2x + c$ and these are the solutions to the differential equation (one solution for each constant, c.

Note, $\int \frac{dy}{dx} dx = \int dy = \int 1dy = y \ (+c)$.

Example 4.17 Solve the differential equation $\frac{dy}{dx} = e^x - \sin x + 2\cos 2x$.

Integrating both sides with respect to x gives, $\int dy = \int e^x - \sin x + 2\cos 2x \ dx$

$\Rightarrow \quad y = e^x + \cos x + \sin 2x + c$

Test 4.15 Solve the differential equation $\frac{dy}{dx} = \frac{1}{x} + \ln x + e^{-x}$

Differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{f(x)}{F(y)}$ are also separable. We write them in the form $F(y)\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)$ and then integrate.

Example 4.18 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{1+\sin y}$.

We write this as $(1+\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}=x$. Integrating gives,

$$\int(1+\sin y)\,\mathrm{d}y=\int x\,\mathrm{d}x \quad\Rightarrow\quad y-\cos y=\frac{1}{2}x^2+c \text{ or } 2y-2\cos y=x^2+c.$$

Example 4.19 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{1+x}$.

We write this in the form $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x}$. Integrating both sides with respect to x gives,

$$\int\frac{1}{y}\mathrm{d}y=\int\frac{1}{1+x}\,\mathrm{d}x \quad\Rightarrow\quad \ln y=\ln(1+x)+c.$$ We can write $c=\ln K$, to give,

$$\ln y=\ln|1+x|+\ln K=\ln\left(K(1+x)\right) \quad\Rightarrow\quad y=K(1+x).$$

Example 4.20 Solve the differential equation $\frac{\sin x}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$.

First, we need to separate the variables, i.e. take all the terms involving x to the LHS and all the terms involving y to the RHS,

$\frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos x}{\sin x}$. Integrating both sides with respect to x gives,

$$\int\frac{1}{1+y}\mathrm{d}y=\int\frac{\cos x}{\sin x}\,\mathrm{d}x \quad\Rightarrow\quad \ln(1+y)=\ln(\sin x)+c=\ln(\sin x)+\ln K$$

$$\Rightarrow\quad 1+y=K\sin x \quad\Rightarrow\quad y=K\sin(x)-1$$

Note, the integral $\int \frac{\cos x}{\sin x}\,\mathrm{d}x$ is of the form $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ (see section 3.5).

Test 4.16 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = (y+2)(x+1)$

Test 4.17 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - y$

Implicit Differentiation

Most of the time, we write functions down in the form $y = f(x)$ with *y* on the LHS and all terms involving *x* on the RHS. Sometimes, this is not possible, for example we cannot write the function $\ln y - \sin x = xe^y$ in the form $y = f(x)$. This is an example of an *implicit function*. Another example of an implicit function, which should seem familiar, is $x^2 + y^2 = 25$. So how do we differentiate implicit functions?

Example 4.21 Given that $x^2 + y^2 = 25$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differentiating term by term with respect to *x*, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(25).$$

How do we calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)$? Recall the chain rule (see section 3.4). We can say that,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

So we have, $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Rearranging for $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Example 4.22 Given that $\sin x+\cos y=e^x+\ln y$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differentiating term by term with respect to *x*, we get,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x+\frac{\mathrm{d}}{\mathrm{d}x}\cos y=\frac{\mathrm{d}}{\mathrm{d}x}e^x+\frac{\mathrm{d}}{\mathrm{d}x}\ln y \quad\Leftrightarrow\quad \cos x+\frac{\mathrm{d}}{\mathrm{d}y}(\cos y)\frac{\mathrm{d}y}{\mathrm{d}x}=e^x+\frac{\mathrm{d}}{\mathrm{d}y}(\ln y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow\quad \cos x-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=e^x+\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$. Rearranging for $\frac{\mathrm{d}y}{\mathrm{d}x}$,

$$\cos x-e^x=\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}+\sin y\frac{\mathrm{d}y}{\mathrm{d}x} \quad\Rightarrow\quad \cos x-e^x=\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{y}+\sin y\right)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{\cos x-e^x}{\frac{1}{y}+\sin y}$$

Example 4.23 If $x^2+y^2-2x-6y+5=0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Differentiating term by term with respect to *x*, we get,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2)+\frac{\mathrm{d}}{\mathrm{d}y}(y^2)\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{\mathrm{d}}{\mathrm{d}x}(2x)-\frac{\mathrm{d}}{\mathrm{d}y}(6y)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x}(5)=\frac{\mathrm{d}}{\mathrm{d}x}(0)$$

$$\Rightarrow\quad 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}-2-6\frac{\mathrm{d}y}{\mathrm{d}x}=0$$. Rearranging for $\frac{\mathrm{d}y}{\mathrm{d}x}$,

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}(2y-6)=2-2x \quad\Rightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-x}{y-3}.$$

Now, to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-x}{y-3}\right)$, i.e. we need to differentiate a quotient. To do this, we simply use the quotient rule (section 3.4), but remember, when we differentiate a function of *y* with respect to *x*, we differentiate the function with respect to *y* and then multiply by $\frac{\mathrm{d}y}{\mathrm{d}x}$, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}f(y)=\frac{\mathrm{d}}{\mathrm{d}y}f(y)\frac{\mathrm{d}y}{\mathrm{d}x}$. Using the quotient rule, then we have,

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-x}{y-3}\right)=\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(1-x)\right)(y-3)-\left(\frac{\mathrm{d}}{\mathrm{d}x}(y-3)\right)(1-x)}{(y-3)^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-x}{y-3}\right)=\frac{-1(y-3)-\frac{\mathrm{d}y}{\mathrm{d}x}(1-x)}{(y-3)^2}.$$

Now, we have found earlier that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-x}{y-3}$. Substituting this in gives,

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-x}{y-3}\right)=\frac{(3-y)-\frac{(1-x)^2}{y-3}}{(y-3)^2}=\frac{(y-3)(3-y)-(1-x)^2}{(y-3)^3}.$$ So, we have the answer,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(y-3)(3-y)-(1-x)^2}{(y-3)^3}.$$

Test 4.18 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x^3+y^3-3y^2\sin x=8$ *Hint*: use the product rule, be careful when differentiating y terms

Parametric Differentiation

Recall the material from section 4.2 on parametric equations. Differentiating a curve defined parametrically is not difficult, we simple need to recall the chain rule.

Example 4.24 A curve is defined parametrically by $x=\frac{1}{1+t}$ and $y=\frac{t}{1-t}$.

Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

From the equation $x=\frac{1}{1+t}$, we can see that $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{(1+t)^2}$. From the equation $y=\frac{t}{1-t}$, using the quotient rule we can see that $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{(1-t)-t(-1)}{(1-t)^2}=\frac{1}{(1-t)^2}$.

From the chain rule, we can say that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x}$. We also recall that $\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$, so we can write $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}.\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

Hence, we have that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(1-t)^2}\times\frac{1}{-\frac{1}{(1+t)^2}}=-\left(\frac{1+t}{1-t}\right)^2$.

So the answer is, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{1+t}{1-t}\right)^2$.

Example 4.25 A curve is defined parametrically by $x=\sin t$ and $y=-\cos t$. Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

To calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$, we follow the same method as example 4.24.

From the equation $x=\sin t$, we can see that $\frac{\mathrm{d}x}{\mathrm{d}t}=\cos t$. From the equation $y=-\cos t$, we can see that $\frac{\mathrm{d}y}{\mathrm{d}t}=\sin t$.

We have, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}.\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\sin t}{\cos t}=\tan t$. So we have found that $\frac{\mathrm{d}y}{\mathrm{d}x}=\tan t$.

Now, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(\tan t)$.

So, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}(\tan t)\frac{\mathrm{d}t}{\mathrm{d}x}$

$$= \sec^2 t \frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 t . \frac{1}{\cos t}$$

$$= \frac{\cos t}{\sin^2 t} = \frac{\cot t}{\sin t}$$. So we have the answer, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\cot t}{\sin t}$.

Test 4.19 A curve is defined parametrically by $x = \frac{1}{1+\sqrt{t}}$ and $y = \frac{1}{1-\sqrt{t}}$.

Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Integration Using Partial Fractions

In section 3.4, we stated a general rule for differentiating quotients. Unfortunately, there is no general rule for integrating quotients, but we can use some of the methods that we have studied earlier to make some progress.

We can integrate functions of the form $\frac{a}{(bx+c)^n}$, and we have learned how to break down some more complicated quotients into a sum of fractions of the form $\frac{a}{(bx+c)^n}$ (partial fractions), which we can integrate. So, splitting complicated quotients into a sum of simpler quotients which we can integrate is the way in which we shall proceed.

Example 4.26 Evaluate $\int \frac{7x+3}{(x-1)(2x+3)} \mathrm{d}x$.

As we have said, there is no general rule that we can use to evaluate this integral directly. For problems of this sort, we need to try to break down the integrand into a simpler form using partial fractions and hope that we can integrate this simpler form. In example 4.5, we found that,

$$\frac{7x+3}{(x-1)(2x+3)} \equiv \frac{2}{x-1} + \frac{3}{2x-3}.$$

So, we can write $\int \frac{7x+3}{(x-1)(2x+3)} \mathrm{d}x = \int \frac{2}{x-1} + \frac{3}{2x-3} \mathrm{d}x$

(Remember that, in general, $\int \frac{a}{bx+c}\,\mathrm{d}x = \frac{a}{b}\ln(bx+c)+K$)

Now, $\int \frac{2}{x-1}+\frac{3}{2x-3}\,\mathrm{d}x = 2\ln(x-1)+\frac{3}{2}\ln(2x-3)+c$.

So, $\int \frac{7x+3}{(x-1)(2x+3)}\,\mathrm{d}x = 2\ln(x-1)+\frac{3}{2}\ln(2x-3)+c$.

Example 4.27 Evaluate $\int \frac{-11x-19}{(2x+3)(x-1)}\,\mathrm{d}x$.

From example 4.6, we found that $\frac{-11x-19}{(2x+3)(x-1)} \equiv \frac{1}{2x+3}-\frac{6}{x-1}$.

So, $\int \frac{-11x-19}{(2x+3)(x-1)}\,\mathrm{d}x = \int \frac{1}{2x+3}-\frac{6}{x-1}\,\mathrm{d}x = \frac{1}{2}\ln(2x+3)-6\ln(x-1)+c$.

Hence, we have the solution $\int \frac{-11x-19}{(2x+3)(x-1)}\,\mathrm{d}x = \frac{1}{2}\ln(2x+3)-6\ln(x-1)+c$.

Test 4.20 Evaluate $\int \frac{(x+1)}{(3x-4)(x+3)^2}\,\mathrm{d}x$

Test 4.21 Evaluate $\int \frac{x^2}{(x+5)(x-3)}\,\mathrm{d}x$

4.6 Vectors

Introduction

A ***vector*** is a mathematical object which has both magnitude and direction. Vector quantities occur commonly in applied maths and physics, for example force, velocity and acceleration are examples of vector quantities – they have a numerical value and a direction. Physical quantities which are not vectors, i.e. they have a magnitude but no direction, are called ***scalars***. Length, area, mass, temperature, energy are examples of scalar quantities. We can represent physical vector quantities such as force and velocity by straight lines in 2 or 3 dimensions, where the length of the line represents the magnitude of the vector and the direction of the line indicates the direction in which the vector quantity is acting (we use arrows to represent the direction of the vector). For example, a force of 10N acting horizontally to the right can be represented by a straight horizontal line of a certain length (for example, we could choose a scale for our diagram of 1N = 1cm),

A force of 20N acting horizontally to the right can be represented by a straight horizontal line of twice the length of the precious vector (using the same scale as before),

Consider a 2-dimensional vector joining points O (the origin) and A as shown below.

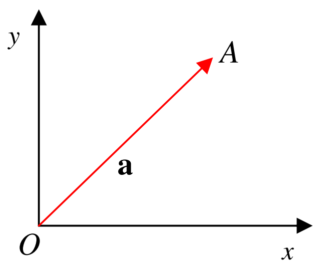


fig. 4.5

We denote this vector using the symbol $\overrightarrow{OA}$ (or $\overrightarrow{OA}$, alternatively we can give it a name, such as $\mathbf{a} = \overrightarrow{OA}$ (when vectors are denoted in this way, **bold** letters are always used). The vector $\mathbf{a} = \overrightarrow{OA}$ is called the *position vector* of point A.

Suppose that the point A has coordinates $A = (1,\ 2)$, then we can define the vector **a** (or $\overrightarrow{OA}$) as a ***column vector***, $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This tells us that the vector **a** is equivalent to a vector of magnitude one pointing in the positive x-direction followed by a vector of magnitude two pointing in the positive y-direction, as shown below.

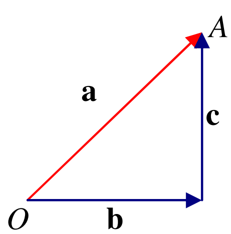


fig. 4.6

This is equivalent to saying that if we want to travel from point O to point A, we can either travel directly along vector **a**, or alternatively, we can travel along vector **b** and then travel along vector **c**. Equivalently, starting at the origin, we could travel along vector **c** first and then travel along vector **b**, this will still take us from O to A, as shown in *fig. 4.7*.

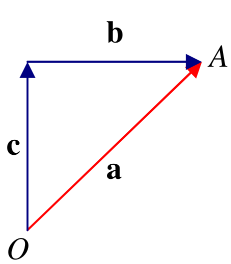


fig. 4.7

Vector **b** is a vector of magnitude one in the positive x-direction, so it can be written as $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (the zero indicates that there is no component of this vector in the y-direction). Vector **c** is a vector of magnitude two in the positive y-direction, so it can be written as $\mathbf{c} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (the zero indicates that there is no component of this vector in the x-direction).

Adding Vectors

To add two vectors together, we simply add together the x components of the two vectors together and add the y components of the two vectors together. For example with the vectors as defined above,

$$\mathbf{b}+\mathbf{c}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}1+0\\0+2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}=\mathbf{a}\,.$$

So, as we have already noticed, $\overrightarrow{OA}=\mathbf{a}=\mathbf{b}+\mathbf{c}=\mathbf{c}+\mathbf{b}$, as illustrated in *fig. 4.6* and *fig. 4.7*.

We can subtract vectors in a similar way, for example if $\mathbf{a}=\begin{pmatrix}7\\-1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}9\\4\end{pmatrix}$, then

$$a-b=\begin{pmatrix}7-9\\-1-4\end{pmatrix}=\begin{pmatrix}-2\\-5\end{pmatrix}.$$

Multiplying a Vector by a Scalar

Consider the vector $\mathbf{a}=\begin{pmatrix}4\\-2\end{pmatrix}$, as shown in *fig. 4.8*.

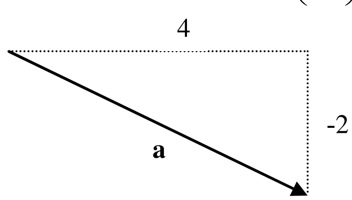


fig. 4.8

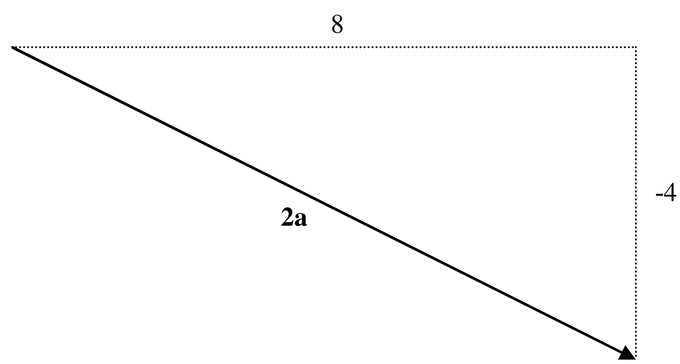


fig. 4.9

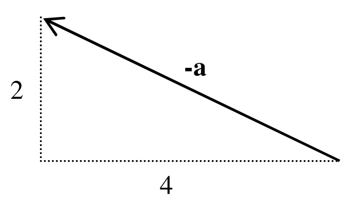


fig. 4.10

Multiplying a vector by a scalar (a number) maintains the direction of the vector but changes the magnitude of the vector, provided the scalar is positive. If we multiply a vector by a negative scalar, then we will get a vector which points in the opposite direction to the original. For example if we multiply **a** in *fig. 4.8* by 2, we get a vector which points in the same direction as **a** but has twice the magnitude of **a**. We have

$2\mathbf{a} = 2\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2\times 4 \\ 2\times -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$, as shown in *fig. 4.9*. If we multiply **a** by -1, we get a vector which is equal in magnitude to **a** but points in the opposite direction. We have $\mathbf{-a} = -1\times\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -1\times 4 \\ -1\times -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, as shown in *fig. 4.10*. Multiplying **a** by -3, for example, would produce a vector pointing in the same direction as –**a**, but with three times the magnitude of –**a**.

Example 4.27 If $\mathbf{a} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, calculate

a) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ **b)** $\mathbf{a}-\mathbf{b}+\mathbf{c}$ **c)** $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

a) $\mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix} -3+2+3 \\ 1+7+(-4) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b)** $\mathbf{a}-\mathbf{b}+\mathbf{c} = \begin{pmatrix} -3-2+3 \\ 1-7+(-4) \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \end{pmatrix}$

c) $2\mathbf{a}-\mathbf{b}+3\mathbf{c} = \begin{pmatrix} 2\times(-3)-2+3\times 3 \\ 2\times 1-7+3\times(-4) \end{pmatrix} = \begin{pmatrix} 1 \\ -17 \end{pmatrix}$.

Test 4.22 Sketch the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ 'head to tail' (vectors **b** and **c** are drawn 'head to tail' in *fig. 4.6*.) On this diagram, draw in the vector $\mathbf{c} = \mathbf{a}+\mathbf{b}$. If $c = \begin{pmatrix} m \\ n \end{pmatrix}$, what are the values of *m* and *n*. Also sketch a similar diagram to show illustrate the calculation $\mathbf{d} = \mathbf{a} - \mathbf{b}$ (*hint*: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$). If $c = \begin{pmatrix} p \\ q \end{pmatrix}$, what are the values of *p* and *q*.

Magnitude of a vector

The *magnitude*, *modulus* or *length* of a vector is calculated using Pythagoras' Theorem. The magnitude of a vector, **a**, is usually written as $|\mathbf{a}|$. For example look back at the vector **a** in *fig. 4.8*. The magnitude of this vector, by Pythagoras is,

$$|\mathbf{a}| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}\,.$$

The magnitude of a vector is **always positive**. For example, the magnitude of vector –**a** in *fig. 4.10* is also $2\sqrt{5}$.

Often we work with vectors in 3-dimensions. To calculate the length of a vector in 3-dimensions we simply use the familiar 3-dimensional version of Pythagoras' Theorem. For example the length of the vector $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$ is $|\mathbf{r}| = \sqrt{3^2 + 1^2 + (-5)^2} = \sqrt{35}$.

Parallel vectors

It is intuitively obvious what we mean by parallel vectors. Perhaps the only point to note is that, for example, vectors **a** and –**a** as in *fig. 4.8* and *4.10* are parallel even though they are travelling in opposite directions (so we cannot define parallel vectors as 'vectors which travel in the same direction'). Vector 2**a** is also parallel to vector **a** (and vector –**a**). In fact, any scalar multiple of vector **a** is parallel to vector **a**, i.e. vector $\lambda\mathbf{a}$ is parallel to vector **a** where λ is any scalar (positive or negative). For example, the vector $\begin{pmatrix} 24 \\ 48 \end{pmatrix}$ is parallel to vector **a**.

Position vectors

Consider the origin, *O* and a point in the 2-dimensional plane, *P*. When working with 2-dimensional vectors, we often use the symbol **i** to denote a unit vector in the *x*-direction, that is a 'horizontal' vector pointing in the positive *x*-direction of length one, as shown in *fig. 4.11*. Similarly we use the symbol **j** to denote a 'vertical' unit vector pointing in the positive *y*-direction. Suppose point *P* is located (with respect to the origin, *O*) *a* units in the **i** direction and *b* units in the **j** direction. Then we can write the vector joining the origin to the point *P* as

$$\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j}.$$

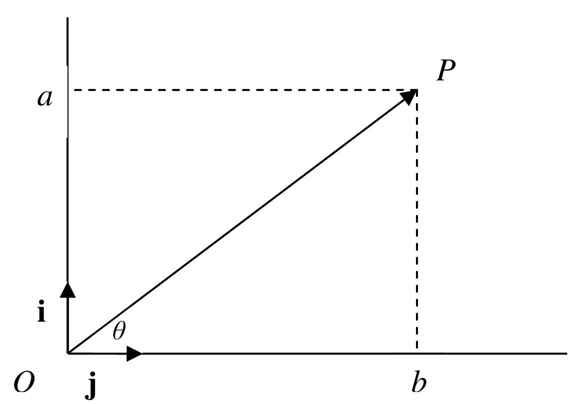


fig. 4.11

The distance from point *P* to the origin is calculated using Pythagoras,

$$\left|\overrightarrow{OP}\right| = \sqrt{\left(a^2 + b^2\right)}.$$

We can work out the direction of the vector $\overrightarrow{OP}$ by using the tan function,

$\theta = \tan^{-1}\left(\frac{a}{b}\right)$.

Vector equations of lines

Let us consider how to write the equation of a straight line in vector form. We are already familiar with writing the equation of a straight line in two dimensions in cartesian form, $y = mx + c$, where m is the gradient of the line and c is the intercept on the y-axis, and maybe we have worked with cartesian equations of lines in three dimensions, where we will have three variables, x, y and z. Notice that, to uniquely specify a straight line in two dimensions we need two pieces of information, for example in the cartesian form we know the gradient and the intercept. How can we uniquely specify a straight line in three dimensions? When we need to find the vector equation of a line, we always look to find two pieces of information: *the position vector of a point on the line* and *any vector parallel to the line*. You will always need to remember these two key pieces of information when working with vector equations of lines. Remember, the position vector of a point on the line, say point A is the vector which joins the origin to the point A. Look at *fig. 4.12*. The vectors in *fig. 4.12* may represent vectors in two or three dimensions. To specify line **r**, we start at the origin and first move along vector **a** to a point on the line, point A. Once we are at point A, there are many lines which pass through that point, so we need another piece of information to uniquely determine line **r**, the extra piece of information we look for is a vector parallel to the line, vector **b**. Once we are at point A, we move in the direction of vector **b** and we are now travelling alone line **r**. The scalar parameter λ just stands for the distance which we move along the line from point A. The vector equation of line **r** is

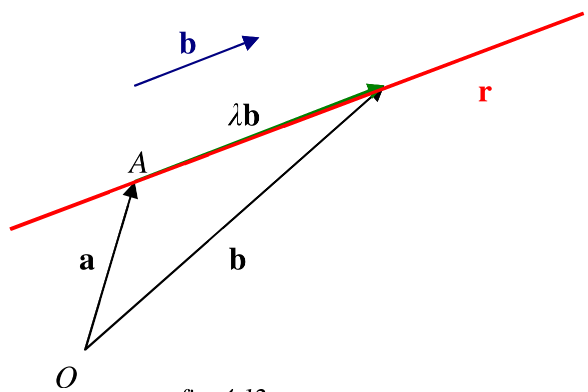


fig. 4.12

$$r = \mathbf{a} + \lambda\mathbf{b}$$

Where **a** is the position vector of a point on the line and **b** is any vector parallel to the line.

Example 4.28 Find the vector equation of the line that passes through the points $(2,1,3)$ and $(5,6,1)$.

We have that $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ is the position vector of a point on the line. To find a vector parallel to the line we simply subtract the two vectors. A vector parallel to the line is

$\begin{pmatrix}5\\6\\1\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}3\\5\\-2\end{pmatrix}$. We have therefore that the vector equation of the line is

$\mathbf{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\5\\-2\end{pmatrix}$.

Converting between cartesian and vector form

Consider the previous example. We can rewrite the vector equation of this line in component form as,

$x=2+3\lambda \qquad y=1+5\lambda \qquad z=3-2\lambda$.

We can rearrange these expressions for λ,

$\lambda=\dfrac{x-2}{3} \qquad \lambda=\dfrac{y-1}{5} \qquad \lambda=\dfrac{z+3}{2}$.

Since the above expressions for λ are all equal, we may write,

$$\frac{x-2}{3}=\frac{y-1}{5}=\frac{z+3}{2}$$

Which is the cartesian form of the line.

Example 4.29 Find the vector form for the line with cartesian equation $y=3x-4$.

As always, we are looking for a point on the line and a vector parallel to the line. One obvious point which lies on the line is the *y*-intercept, which has position vector $\begin{pmatrix}0\\-4\end{pmatrix}$. The gradient of the line is 3. An obvious vector which is parallel to the line (has gradient 3) is $\begin{pmatrix}1\\3\end{pmatrix}$. The vector equation of the line is therefore $\mathbf{r}=\begin{pmatrix}0\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\3\end{pmatrix}$.

Test 4.23 Find the vector equation of the line that passes through the points $(5,5,1)$ and $(-2,1,6)$. Also write the equation of the line in cartesian form.

Intersection of two lines

In two dimensions, straight lines that are not parallel will meet at a point (they cross). In three dimensions, it is rarer to have two lines that meet. In three dimensions, if two lines do not cross they are called *skew*.

Example 4.30 Do the lines $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ meet? If so, find the point of intersection.

First we write each of the lines $\mathbf{r}_1$ and $\mathbf{r}_2$ in component form.

For $\mathbf{r}_1$ we have:

$$\begin{aligned} x &= 3+\lambda \\ y &= 4-3\lambda \end{aligned} \qquad \text{.......(1)}$$

For $\mathbf{r}_2$ we have:

$$\begin{aligned} x &= 3+\mu \\ y &= -2+4\mu \end{aligned} \qquad \text{.......(2)}$$

If the lines are to cross, we need the *x* value in (1) to equal the *x* value in (2), ie. We need,

$$3+\lambda = 3+\mu \quad \Leftrightarrow \quad \lambda = \mu .$$

We also need the *y* value in (1) to equal the *y* value in (2), ie. We need,

$$4-3\lambda = -2+4\mu .$$

But since $\lambda = \mu$ we can write the above line as,

$$4-3\lambda = -2+4\lambda \quad \Rightarrow \quad \lambda = \frac{6}{7} = \mu .$$

Substituting the value of λ or μ into (1) or (2) gives $x = 3\frac{6}{7}$ and $y = 2\frac{4}{7}$. Hence the lines meet at the point $\left(3\frac{6}{7},\ 2\frac{4}{7}\right)$.

Test 4.24 Do the lines $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ meet? If so, find the point of intersection.

Scalar product

The scalar product (or dot product) of two vectors is calculated as follows:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} d \\ e \\ f \end{pmatrix} = a \times d + b \times e + c \times f \ .$$

Note, we write a 'dot' between the two vectors to denote that we are taking the scalar product. Let us do a numerical example. If $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$, then

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = 1 \times 2 + 4 \times 2 + (-3) \times 6 = -8 \, .$$

We use the following useful formula to calculate the angle between two vectors, θ as shown in *fig. 4.13.*:

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

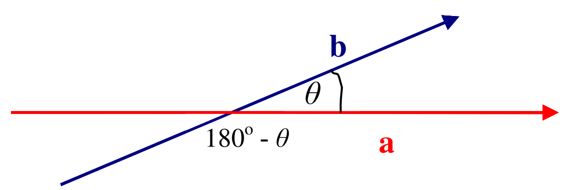


fig. 4.13

Which we can rearrange as:

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Notice that if two lines are perpendicular, $\theta = 90^\circ$ and so $\cos\theta = 0$ and so $\mathbf{a}.\mathbf{b} = 0$.

Example 4.31 Let us continue from example 4.30. We found that the two lines meet at the point $\left(3\frac{6}{7},\ 2\frac{4}{7}\right)$. What is the angle between the two lines at this point? We use the scalar product formula. The directional vector of the line $\mathbf{r}_1$ is $\begin{pmatrix}1\\-3\end{pmatrix}$ (the vector $\begin{pmatrix}3\\4\end{pmatrix}$ is the position vector of a point on the line). So, in the scalar product formula, we will have $\mathbf{a}=\begin{pmatrix}1\\-3\end{pmatrix}$ (which comes from line $\mathbf{r}_1$). Similarly, the directional vector of the line $\mathbf{r}_2$ is $\begin{pmatrix}1\\4\end{pmatrix}$, so in the scalar product formula, we will have $\mathbf{b}=\begin{pmatrix}1\\4\end{pmatrix}$. We find the scalar product of these two vectors:

$$\mathbf{a}.\mathbf{b}=\begin{pmatrix}1\\-3\end{pmatrix}.\begin{pmatrix}1\\4\end{pmatrix}=1-12=-11.$$

We find the length of each of these vectors:

$$|\mathbf{a}|=\sqrt{1^2+(-3)^2}=\sqrt{10}$$

$$|\mathbf{b}|=\sqrt{1^2+4^2}=\sqrt{17}.$$

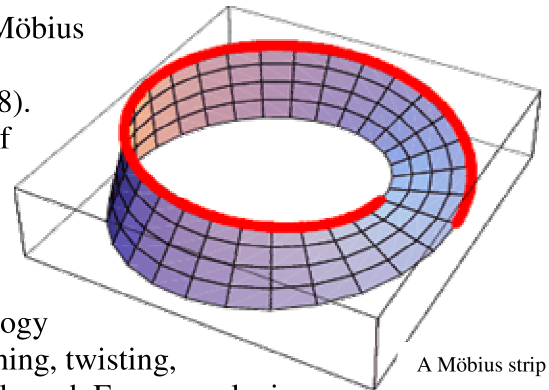
So, from the scalar product formula, we have:

$$\cos\theta=\frac{-11}{\sqrt{10}\sqrt{17}}=-0.8437 \quad \Rightarrow \quad \theta=147.5^\circ.$$

So, the angle between the two vectors is 147.5°.

Test 4.24 Do the lines $\mathbf{r}_1=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}-2\\4\\-1\end{pmatrix}$ and $\mathbf{r}_2=\begin{pmatrix}3\\-5\\-5\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\-1\end{pmatrix}$ meet? If so, find the point of intersection and the angle between the two vectors.

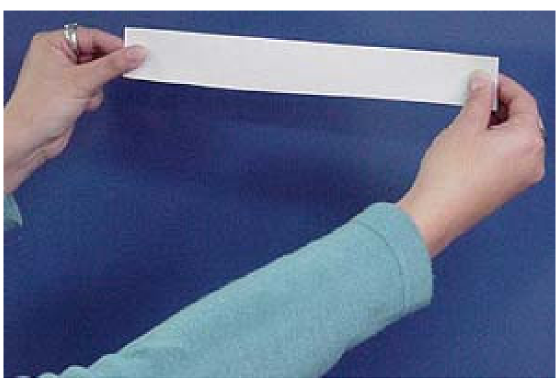
Information about the cover – Möbius strip


A Möbius strip

The picture on the cover shows a Möbius strip (or a Möbius band) named after the German mathematician who discovered it, August Ferdinand Möbius (1790 – 1868). This is a curious object which arises from the study of a branch of mathematics called topology. Topology is the study of spatial objects such as curves, surfaces and the spacetime of general relativity. Topology is sometimes informally referred to as 'rubber sheet geometry' because in the study of topology spatial objects are considered equivalent under stretching, twisting, deformations. Tearing and gluing, however are not allowed. For example, in topology a circle and an ellipsoid are equivalent.

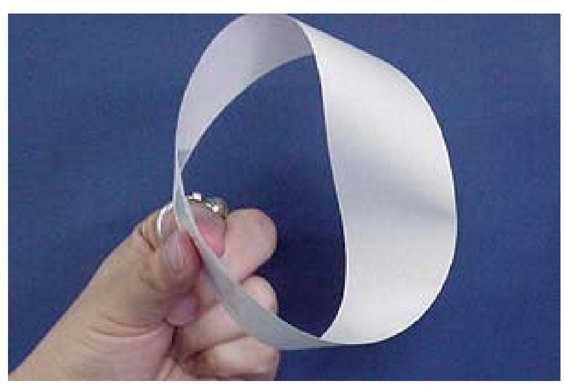
The Möbius strip is a curious object because it has only one side. You can understand more about the Möbius strip by make one.

Step 1: Take a strip of paper
Step 2: Hold the paper at each end and twist 180 degrees
Step 3: Attach the ends of the strip together
Step 4: You now have a completed Möbius strip


Step 1


Step 2


Step 3


Step 4

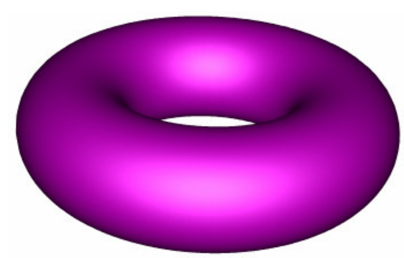
Now, to see that the Möbius strip has only one side, take a pencil and draw a line alone one side of the Möbius strip, following the side all the way around until you get back to the beginning. What do you notice? The Möbius strip has only one side!

Another curious thing happes if you try to cut the Möbius strip down the middle. You might expect to get two separate Möbius strip. What actually happens? What happens if you cut the new Möbius strip down the middle?

Cutting the Möbius strip along its length

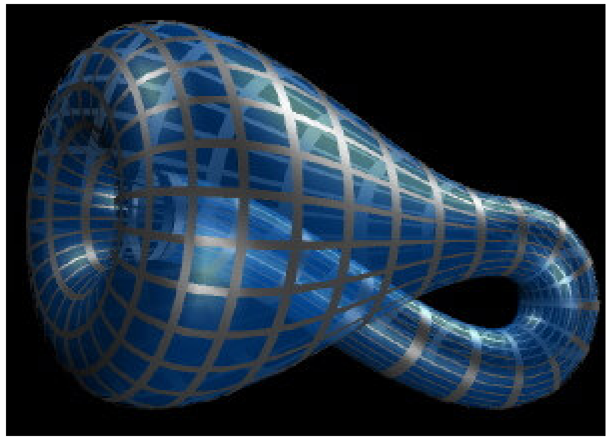
If you have ever been to Blackpool Pleasure Beach and rode on the old wooden rollercoaster 'The Grand National' then you have rode around a Möbius strip! The wooden track of The Grand National roller coaster is actually a Möbius strip. The Grand National roller coaster has a track with two carriages that race side-by-side. At the start of the ride there are two carriages either side of a boarding platform. You will notice that at the end of the ride, you return to the opposite side of the platform to which you started but the tracks do not cross!

The Grand National



A torus

The Möbius strip is related to other topological objects, such as the torus (donut or bagel shape) and the Klein bottle (named after the German mathematician Felix Christian Klein, 1849 – 1925). The torus and the Klein bottle can both be cut in certain ways as to produce Möbius strips (cutting the Klein bottle in half along its length produces two Möbius strips). The Klein bottle itself is a strange topological object. It is a smooth surface that does not end. A fly can move from the outside to the inside without passing through the body of the bottle (this is not true, for example, for a sphere) and so the Klein bottle actually has no outside and no inside! Physically, the Klein bottle can only actually be realised in four dimensions since it passes through itself without the presence of a hole.



A Klein bottle

Topology is a complicated area of pure mathematics. Here I give just a brief flavour of some of the less technical aspects of the Möbius strip and related objects.

August Ferdinand Möbius